The Physics of Guitar Strings

R. R. McNeil

1. Introduction

The guitar makes a wonderful device to demonstrate the physics of waves on a stretched string. This is because almost every student has seen a guitar close up and many have strummed one or experienced first hand the different sounds emitted by plucking the strings on a guitar. Consequently, most students already know something about how a guitar works would be interested to know more about it.

I’d like to pose some questions regarding the guitar before beginning the explanations and demonstrations. Holding up a guitar and plucking a few strings, I came up with this set of questions:

1. Why are the different strings different thicknesses? Why do they sound different?

2. Why are those little metal strips (they are called frets) positioned so irrationally along the neck of the guitar?

3. How do I decide where to put my finger to make the different musical notes?

4. At various places along the string, if I place my finger lightly on the string and pluck the string, an interesting sound which is high in pitch is emitted by the guitar. But not everywhere on the string does this happen; just some places. Where are those places and why? What’s going on that makes those sounds?

I hope to answer these questions and to utilize this wonderful musical instrument as a tool to demonstrate the physics of standing waves on a stretched string.
2. Waves on a String

Waves on a string are transverse waves. This means that the displacement of the string from equilibrium is transverse to the direction of the wave propagation which is along the length of the string (see figure 1).

![Figure 1. Transverse wave along a string under a tension F. The wave travels to the right because of the force which restores the displaced string segment back to its equilibrium position.](image)

The speed of the wave on a string depends on two parameters: the tension in the string and the mass per length or linear mass density of the string. The speed of the wave is dependent on the speed at which the displaced string segment is brought back to equilibrium. The tension in the string is what is responsible for accelerating the displaced string segment back to equilibrium. Therefore, if the tension is increased, the string segment is brought back to equilibrium faster, and the wave travels faster. On the other hand, if the string segment is lighter (lower linear mass density), the tension will be able to produce a greater acceleration of the string segment back to equilibrium and hence the speed of the wave would be increased. The exact relationship for the speed of the wave on a string is given by:

\[ v = \sqrt{\frac{F}{M/L}} \]  

(1)

where F is the tension in the string, M is its mass, and L is its length. The ratio M/L is the linear mass density of the string. The frequency \( f \) in cycles/sec or Hertz (Hz) and wavelength \( \lambda \) in units of length of any wave depends on the speed of the wave and follows the relation:

\[ v = f\lambda \]  

(2)

A wave which travels down a string is a traveling wave and when such a wave hits a fixed end, the wave reverses direction and travels back. This is called reflection. However,
the wave is inverted when it reflects off the fixed end (see figure 2). Try this out yourselves!

![Figure 2. Transverse wave along a string reflects from a fixed end and inverted upon reflection.](image)

If two or more traveling waves are moving through a medium, the resultant wave is found by adding the displacements of the individual waves together. This is called the **Principle of Superposition**. If the string is fixed at both ends and has a length equal to an integer number of half-wavelengths of the wave on the string (remember that the wavelength depends on the frequency and the speed of the wave), then the incident wave and reflected wave will combine to form a **standing wave**. The relation then between the frequency of the wave and the speed and length of the string is then given by

\[
f_n = \frac{v}{\lambda_n} = \frac{vn}{2L} = \frac{n}{2L} \sqrt{\frac{F}{M/L}}
\]

where \(n\) is an integer 1, 2, 3, … For example, the lowest frequency which is called the **fundamental frequency** occurs for \(n = 1\) and would occur when the wavelength is equal to \(2L\). This is shown in figure 3.

![Figure 3. Standing wave on a stretched string of length L, fixed at both ends. The displaced standing wave represents the fundamental frequency or first harmonic.](image)

The fundamental frequency results in the middle of the string vibrating up and down with maximum amplitude (see figure 3). This location of maximum amplitude of the transverse displacement is called an **anti-node**. At the two fixed ends the string does not
move up and down. These points are call **nodes**. If the string has a standing wave with twice the frequency of the fundamental, then one complete wavelength would be observed on the string. See figures 4. There would then be a point mid-way along the string where the string displacement would be zero. This point would then be a node, as would still be the points at the two fixed ends. The points mid-way between the mid-point on the string (node) and the fixed ends (node) would be oscillating with maximum amplitudes. These points would be anti-nodes.

![Standing wave diagram](image)

**Figure 4.** Standing wave on a stretched string of length $L$, fixed at both ends. The upper displaced standing wave represents the $n=2$ or second harmonic frequency while the lower wave represents the $n=3$ or third harmonic.

When the wave is driven on the string by a vibrator at a frequency such that a standing wave is produced according to equation 3, the energy of the vibration can be transferred to the wave producing a resonance. The amplitude of vibration is related to the energy carried by the wave. Only at the specific frequencies given by equation 3 can the string resonate. When you pluck a stretched string, depending on where you pluck, you are exciting many of the characteristic frequencies. However you are mostly passing energy into the fundamental mode (frequency). So, a plucked string on a musical instrument is predominantly emitting the pitch of the fundamental frequency; the other frequencies of the harmonics are present, just quieter. However, you can suppress the fundamental frequency by placing your finger on the string where the fundamental frequency has an anti-node (e.g. the mid-point).
3. Guitar strings

There are six strings on a standard guitar that are stretched to produce standing waves where the fundamental frequency and corresponding musical note are:

- Sixth string: E (82Hz)
- Fifth string: A (110Hz)
- Fourth string: D (147 Hz)
- Third string: G (196 Hz)
- Second string: B (247 Hz)
- First string: E (330Hz)

The higher the frequency the higher the pitch. The thicker strings have higher linear mass density (mass per length) and so should have a slower speed of the wave on the string. The slower speed results in a lower fundamental frequency. The thick strings sound lower in pitch than the thin string. Here’s an exercise: Weigh and measure the length of the strings to determine the linear mass densities of the strings on a guitar. Compare the values. If the strings are to be stretched between two points equal in distance. Determine then the tension on the string necessary to produce the fundamental frequency corresponding to the frequencies above. Are the strings at the same tension on a guitar? You’ll maybe notice that some guitars have steel first, second and third strings and some guitars have nylon strings. Are the steel strings the same radii as the nylon strings? Do you expect them to be? Why not?

If one stretches the string and puts more tension on it, the pitch of the sound rises. The fundamental frequency will increase as the tension in the string increases. Thus, tuning of the guitar string is accomplished by stretching the string and making the tension necessary to produce the correct frequency. The pitch of the string is usually compared with a tuning fork in setting the tension. Let’s consider one of the guitar strings. The B string. That’s the second to the lightest string. How long this string is depends on the particular guitar. On my current guitar the string is stretched between to fixed points separated by 65.3 cm. Thus \( L_B = 65.3 \text{ cm} \). If the B string is properly tuned, it would resonate at 247 Hz. That means that the fundamental frequency of that string, stretched to 65.3 cm is

\[
f_B = \frac{v_B}{2L_B} = 247 \text{ Hz}
\]

The speed of the wave on the string is then \( v = 2L_B \cdot f_B = 2 \times 0.653 \text{m} \times 247 \text{ 1/s} = 322 \text{ m/s} \). This will be a constant for that string as long as the tension or the linear mass density doesn’t change.
How then can I make different musical notes on the string? Answer: I can make the string shorter. I do that by placing my fingers down on the string to make the distance between my finger (pressed on the string) and the other fixed end of the string less than 65.3 cm. Do you ever wonder what those little metal strips which are placed on the length of the neck of the guitar are for? Those little metal strips are called frets. Why they don’t seem to have any logical reason in there placement? Here’s why:

The different musical notes have different frequencies. Here is a partial list starting for the frequency of the fundamental for the B string on a guitar:

<table>
<thead>
<tr>
<th>Note</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>247Hz</td>
</tr>
<tr>
<td>C</td>
<td>262Hz</td>
</tr>
<tr>
<td>D</td>
<td>292Hz</td>
</tr>
<tr>
<td>E</td>
<td>330Hz</td>
</tr>
<tr>
<td>F</td>
<td>349Hz</td>
</tr>
<tr>
<td>G</td>
<td>392Hz</td>
</tr>
<tr>
<td>A</td>
<td>440Hz</td>
</tr>
<tr>
<td>B</td>
<td>494Hz</td>
</tr>
<tr>
<td>C</td>
<td>523Hz</td>
</tr>
</tbody>
</table>

You’ll notice that going from B to B (climbing up one octave) the frequency actually doubles (247Hz to 494Hz). Knowing that you can work upward and downward in frequency to produce many octaves worth of notes and their corresponding frequencies.

Anyway, if I wish to make a string, which has a fundamental frequency corresponding to one of the above frequencies, to emit a sound with the pitch corresponding to a particular musical note, I have merely to adjust the length of my already stretched string to the desired value by pressing my finger down on the string

\[ f_{\text{Note}} = \frac{v}{2L_{\text{Note}}} \]  \hspace{1cm} (5)

Since the speed of the wave is the same, using equation 4 and 5 a relationship between the length of the string, and the frequency note and the frequency and length of the entire B string can be derived:

\[ L_{\text{Note}} = L_{B} \frac{f_{\text{B}}}{f_{\text{Note}}} \]  \hspace{1cm} (6)
A complete table of the frequencies and the corresponding string length for the B string is given below:

<table>
<thead>
<tr>
<th>Note</th>
<th>Frequency</th>
<th>( L_{\text{Note}}/L_B )</th>
<th>( L_{\text{Note}} ) (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>247Hz</td>
<td>1</td>
<td>65.3</td>
</tr>
<tr>
<td>C</td>
<td>262Hz</td>
<td>0.943</td>
<td>61.6</td>
</tr>
<tr>
<td>D</td>
<td>292Hz</td>
<td>0.840</td>
<td>54.9</td>
</tr>
<tr>
<td>E</td>
<td>330Hz</td>
<td>0.748</td>
<td>48.8</td>
</tr>
<tr>
<td>F</td>
<td>349Hz</td>
<td>0.708</td>
<td>46.2</td>
</tr>
<tr>
<td>G</td>
<td>392Hz</td>
<td>0.630</td>
<td>41.1</td>
</tr>
<tr>
<td>A</td>
<td>440Hz</td>
<td>0.561</td>
<td>36.6</td>
</tr>
<tr>
<td>B</td>
<td>494Hz</td>
<td>0.500</td>
<td>32.7</td>
</tr>
<tr>
<td>C</td>
<td>523Hz</td>
<td>0.472</td>
<td>30.8</td>
</tr>
</tbody>
</table>

You’ll notice that one of the frets is always located at just the right spot\(^1\). They help make the place to put your fingers easier to locate and also provide for exact string lengths for the particular notes. The guitar maker must have known a lot of physics to put them at the correct location! Now, I have taught you to play the guitar! All that you need is a ruler and a calculator. Good luck!

For advanced guitar players, you can suppress the fundamental frequency by holding your finger lightly on the string at the location of the node for the second, third, fourth, etc harmonic. For example, the second harmonic can be selected by putting your finger at the mid-point of the B string (see figure 4). What is the frequency of the second harmonic? It is twice that of the fundamental. You can pluck the string on either side of the finger and get the same note. This you will find you cannot do if you press you finger hard down on the string. You can select the third harmonic by putting your finger lightly on the string at a location 1/3 of the way from either fixed end. What is the frequency of the third harmonic? Answer: 3 times the fundamental: for the B string that would be \( 3 \times 247\text{Hz} = 741\text{ Hz} \). See you at Carnegie Hall!

---

\(^1\) There are more frets than indicate, representing the location the sharps and flats for the notes.