Background Information on Exponentials and Logarithms

Since the treatment of the decay of radioactive nuclei is inextricably linked to the mathematics of exponentials and logarithms, it is important that you have some expertise in using them. This is intended as a brief outline of how to use them. Note that you calculator will have these operations built into it, allowing you to get the result of the operation with a single key stroke.

We might note in passing that the mathematics below is not all limited to the topic in the class, that is radioactive decay. It is not even limited to physics or the other sciences. For example if you establish an account which pays compound interest at a fixed rate, and reinvest the interest into the account, then the value of the account increases exponentially with time, following all of the same mathematical formulae below.

Step 1 - Integral Powers of 10

Let us start with something with which (I hope) you are already familiar, that is powers of 10. The notation 10^x means 10 multiplied by itself x times,

C $10^2 = 10 * 10 = 100$
C $10^3 = 10 * 10 * 10 = 1000$
C $10^6 = 10 * 10 * 10 * 10 * 10 * 10 = 1{,}000{,}000$
C and so on

Step 2 - NonIntegral Powers of 10

Let us take the previous idea on step further, suppose that the power x is not a whole number, say 4.518. Then what is 10^x now? You will need your calculator to solve this one. Depending on your calculator the exact keystrokes will be different, but there is a good chance that there is a key labeled 10^x. Try entering 4.518 followed by 10^x, and you should get the result $10^{4.518} = 32960.97$

Step 3 - Base 10 Logarithms

The previous step solved the equation $y = 10^x$, if you are given x then you can calculate y. If x = 4.518 then y = 32960.97. However, suppose you are given y and are required to calculate x. For example what is the value of x if y = 69. As an equation solve for the variable x if $69 = 10^x$.

The key to solving this equation is the logarithm (abbreviated to just log). It is defined as the opposite operation to the power of 10:

$$\textbf{If } y = 10^x \textbf{ then } x = \log y$$

Again your calculator probably has a key labeled log which performs this operation for you. Type in 69 followed by log, and you should get the result 1.838849. Just a check try using the method above and you should get the result $10^{1.838849} = 69$. (It might not be exactly 69 because the number 1.838849 has been rounded off a little, but it should be very close.)

You might want to try the following for practice:

C solve for the variable x if $6.92 = 10^x$. Answer, $x = 0.8401$
C solve for the variable x if $259.9 = 10^x$. Answer, $x = 2.4148$
C solve for the variable x if $679333 = 10^x$. Answer, $x = 5.8320$
C solve for the variable x if $45.29 = 10^x$. Answer, $x = 1.6560$
C solve for the variable x if $7.14 \times 10^{12} = 10^x$. Answer, $x = 12.8537$
C solve for the variable x if $10000000 = 10^x$. Answer, $x = 7$

Step 4 - Negative powers

In all the examples above x was a positive number, in which case 10^x is always greater than 1. The value of x could be negative, say $x = -0.55$. In that case $y = 10^x$ is less than 1. Your calculator can solve these in the same manner. Type in 0.55, then the change sign key, and then the 10^x key. You should get the result 0.281838.

You might want to try the following for practice:

C If $x = -0.1$ what is $y = 10^x$? Answer, $y = 0.79433$
C If $x = -4$ what is $y = 10^x$? Answer, $y = 0.0001$
C If $x = -2.6754$ what is $y = 10^x$? Answer, $y = 0.00211$
C If $x = -6.3912578$ what is $y = 10^x$? Answer, $y = 0.0000004062021$
C If $x = -9.1$ what is $y = 10^x$? Answer, $y = 7.94 \times 10^{-10}$

Negative powers are handled by your calculator also. Suppose we want to solve $0.01234 = 10^x$. Rearranging the equation using the definition of the logarithm from above, this becomes $x = \log 0.01234$. Try typing in 0.01234 into your calculator and press the log key. You should get the answer $x = -1.90868$

You might want to try the following for practice:

C What is x if $0.75 = 10^x$? Answer, $x = \log 0.75 = -0.12494$
C What is x if $0.0000023 = 10^x$? Answer, $x = \log 0.0000023 = -5.6383$
C What is x if $0.012 = 10^x$? Answer, $x = \log 0.012 = -1.9208$
C What is x if $5.4 \times 10^{-14} = 10^x$? Answer, $x = \log 5.4 \times 10^{-14} = -13.2676$

Step 5 - Dealing with bases other than 10

In all the above expressions we have used base 10, that is $y = 10^x$. However there is nothing special about base 10, except that it is the base that you are used to using, the decimal system. We could have used any base number such as 2 (the binary system) or 8 (octal system) or 16 hexadecimal system, or 11, or anything else

C $2^5 = 2*2*2*2*2 = 32$
C $3^6 = 3*3*3*3*3*3 = 729$
C $16^4 = 16*16*16*16 = 65536$

Again these work even if the exponent is not an integer

- $2^{8.81} = 448.8$ (Your calculator should have a y^x key to perform this operation. Type 2, then y^x, then 8.81, then the = key)
- $16^{9.45} = 239295116727.8$ (Type 16, then y^x, then 9.45, then the = key)
- $3^{-0.45} = 0.610$ (Type 3, then y^x, then 0.45, then the change sign key, then the = key)

Step 6 - Non integral bases

So far we have considered the case when the power is not an integer. Can we do the same if the base is not an integer? The answer is yes.

- $4.234^3 = 4.234*4.234*4.234 = 75.901884904$ (Type 4.234, then y^x, then 3, then the = key)
- $(-0.023)^5 = (-0.023)*(-0.023)*(-0.023)*(-0.023)*(-0.023) = -0.000000006436343$
- $4.123^{7.76} = 59436.8$ (Type 4.123, then y^x, then 7.76, then the = key)

Step 7 - e

One of these non-integer numbers is so special in mathematics it is given its own symbol, e which stands for 2.71828182845904523536028747135 27...... (The reasons why it is so special have to do with calculus and need not concern us here.) However, it in all respects it behaves just like all other bases

- You can raise e to any power. In this case the power is also referred to as an exponent. Your calculator should have a key to do this for you. It is likely labeled e^x or exp, depending on brand.
  - $e^2 = 7.3891$
  - $e^{7.81} = 2465.13$
  - $e^{-5.6} = 0.003698$
  - $e^{0.004} = 1.004008$
- You can take logarithms using e as the base. Note, when taking logarithms above we were using base 10, and the logarithm should more precisely be written as $\log_{10}$ since we could use any base we like. For example in base 4 we would write $\log_4$, and in base 16 we would write $\log_6$. In base e we should write $\log_e$, but since e is special this is given its own designation, ln. Your calculator will likely also have a key labeled ln also.
  - if $5 = e^x$ then $x = \ln(5) = 1.609$. (Type 5 and then the ln key. You should not have to hit the = key to get the answer.)
  - if $6778.3 = e^x$ then $x = \ln(6778.3) = 8.8215$
  - if $0.00045 = e^x$ then $x = \ln(0.00045) = -7.706$
  - if $5.69 \times 10^{12} = e^x$ then $x = \ln(5.69 \times 10^{12}) = 29.370$
  - if $4.12 \times 10^{-31} = e^x$ then $x = \ln(4.12 \times 10^{-31}) = -69.964$

Step 8 - Putting powers and logarithms together

A simple rule is that, *for the same base*, raising the base to a power and finding the logarithm are opposite actions. One followed by the other has no overall effect

- $\log_y(y^x) = x$
  - $\log_{10}(10^4) = 4$
  - $\log_4(4^{9.56}) = 9.56$
  - $\log_e(e^{12}) = \ln(e^{12}) = 12$
  - $\log_e(e^{-0.562}) = \ln(e^{-0.562}) = -0.562$
- $y^{\log_y(x)} = x$
  - $4^{\log_4(12)} = 12$
  - $10^{\log(3.456)} = 3.456$
  - $e^{\ln(2)} = 2$
  - $e^{\ln(0.00045123)} = 0.00045123$

Note though that this rule does not work if the bases are different. For example $\ln(10^3)$ is not three. The power operation used base 10, but the logarithm used base e.

Step 9 - A radioactive half life problem

The equation which describes the radioactive decay of an nucleus is $N = N_o\, e^{-kt}$ where N_o is the initial number of nuclei, N is the number remaining after a time t, and k is a constant related to the lifetime ($k = \ln 2/T = 0.6931/T$). With a little bit of rearranging this can be written as $y = e^x$ if y stands for the fraction N/N_o and x stands for -kt. Using the rules developed above we can solve these radioactive decay problems.

- *Example I* An isotope has a half life of 25 minutes. What fraction remains after one hour?
  - First of all we need to find k. Its value is $\ln 2/T = \ln 2/25 = 0.6931/25 = 0.027726$
  - Putting in t = 60 minutes, we have $x = -kt = -0.027726*60 = -1.66355$
  - Then $y = e^x = e^{-1.66355} = 0.189$ = **18.9 %**
- *Example II* For this same isotope how much would remain the next day
  - Now put t = 24 hours, which is the same as 24*60 = 1440 minutes
  - Then $x = -kt = -0.027726*1440 = -39.9253$
  - The fraction remaining is now $y = e^x = e^{-39.9253}$ = **$4.58 \text{ x} 10^{18}$** , which is a very small fraction. This sample has now almost completely disappeared.
- *Example III* Suppose that you have a sample to be dated using ^{14}C. By measuring the activity of the sample you discover that it has only one third the activity it must have had when it was new. How old is it?
  - First of all calculate k. The half life of ^{14}C is 5730 years, and so $k = \ln 2/T = 0.6931/5730 = 1.20968 \times 10^{-4}$.
  - In this case we know y = **a** = 0.333333, and we need to solve for x knowing that $0.333333 = e^x$.
  - If $y = e^x$ then $x = \ln y = \ln 0.333333 = -1.0986$
  - Since x stands for -kt, to calculate t we have $t = -x/k = -(-1.0986)/1.20968 \times 10^{-4}$ = **9082 years**.

As mentioned above this mathematical treatment relates to similar problems in many walks of life. Let's go through the following examples:

- *Financial Example 1* You put $1000 into an account which bears an annual interest rate of 6% paid monthly, and reinvest the interest into the account. How much is the account worth after 5 years? (Assuming that there are no subsequent deposits or withdrawals.)
  - The monthly interest rate is 6%/12 = ½% = 0.005. After one month the account will have a value of $1000*(1+0.005) = $1000*(1.005), after 2 months $1000*(1.005)*(1.005) = $1000*(1.005)^2, after 3 months $1000*(1.005)^2*(1.005) = $1000*(1.005)^3, and so on. Generalizing this, after x months the value of the account is $1000*(1.005)^x. We have then same mathematics as above ($y = b^x$) if we use a base of 1.005 for the calculations.
  - After 5 years (60 months) the value of the account will be $1000*(1.005)^{60} = $1000*1.34885 = **$1348.85**
- *Financial Example II* How long will it take for the value of the account to rise to $1,000,000?
  - We have to solve the equation $1,000,000 = 1000*1.005^n$. After canceling a factor of 1000 from both sides of the equation this becomes $1000 = 1.005^n$.
  - The solution to this equation (see above) is $n = \log_{1.005} 1000$. If your calculator could do calculations in base 1.005 you could solve this directly, but this is unlikely. Instead we can make use of a useful relationship for logarithms, that $\log_b x = \ln x/\ln b$.
  - For our example $n = \log_{1.005} 1000 = \ln 1000/\ln 1.005 = 6.907755/0.000498754$ = **1385 months or a little over 115 years.**