Multibeam Interference

Introduction

The task is to calculate the result of adding a large number of light beams together to get the resulting interference pattern. Each individual beam is characterized by an oscillating electric field

$$E_j = E_{oj}\, e^{i(kz-\omega t+\varphi)}$$

where E_{oi} is the amplitude, $k=2\pi/\lambda$ is the wavenumber and $\omega=2\pi f$ is the angular frequency. The resultant electric field is the sum of the individual electric fields

$$E = \Sigma_i\, E_j$$

with the intensity of the interference pattern being proportional to the square of the amplitude of the resultant field

Assumptions

To aid in calculating the sum above we shall make some simplifying assumptions. None are restrictive, if they are not fully satisfied, only the details of the final result change.

- That all the individual electric fields are the same, $E_{oj} = E_o$ for all values of j.
- That all the waves have the same frequency and wavelength.
- That the difference in phase between waves j and j+1 is a constant, δ. From that we can write $\varphi_j = (j-1)\,\delta$ relative to the phase of the first beam.

Calculating the Sum

With these assumptions we can write

$$E=\sum_{j=1}^{N} E_o\, e^{i(kz-\omega t+(j-1)\delta)}=E_o\, e^{i(kz-\omega t)}\sum_{j=1}^{N} e^{i(j-1)\delta}$$

The summation is now the geometric sum $1+x+x^2+.\,.\,.\,.\,.+x^{N-1}$, where $x = e^{i\delta}$. The value of the sum is

$$\sum_{j=1}^{N} e^{i(j-1)\delta}=\sum_{j=1}^{N} x^{j-1}=\frac{1-x^N}{1-x}=\frac{1-e^{¿\delta}}{1-e^{i\delta}}$$

Intensity

The intensity of the fringe pattern is then proportional to EE^{*}[1]

1 E^* is the complex conjugate of E.

$$I = E_o^2 \frac{1-e^{iN\delta}}{1-e^{i\delta}} \frac{1-e^{-iN\delta}}{1-e^{-i\delta}} = E_o^2 \frac{2-e^{iN\delta}-e^{-iN\delta}}{2-e^{i\delta}-e^{-i\delta}} = E_o^2 \frac{2-2\cos(N\delta)}{2-2\cos(\delta)}$$

finally a trigonometric transformation[2] gives the final result

$$I = E_o^2 \frac{\sin^2\left(\frac{1}{2}N\delta\right)}{\sin^2\left(\frac{1}{2}\delta\right)}$$

Minima

The intensity of the pattern goes to zero (minima) when the numerator is zero, unless the denominator is also zero. This will occur when

$$\sin\left(\frac{1}{2}N\delta\right) = 0$$

or equivalently when

$$\delta = 2p\pi / N$$

where p is any integer, except for those values where p = qN. For these values $\sin(½\delta) = \sin(q\pi) = 0$ for all q. The expression for the intensity has a zero in both the numerator and denominator.

Maxima

Suppose that δ is close to the value $2q\pi$. Set it equal to $2(q\pi+\varepsilon)$. Then the intensity is

$$I = E_o^2 \frac{\sin^2\left(\frac{1}{2}N\delta\right)}{\sin^2\left(\frac{1}{2}\delta\right)} = E_o^2 \frac{\sin^2(Nq\pi + N\epsilon)}{\sin^2(q\pi+\epsilon)} = E_o^2 \frac{\sin^2(N\epsilon)}{\sin^2(\epsilon)}$$

where the last step makes use of the relationship sin(A+B) = sin(A)cos(B)+cos(B)sin(A). Finally we will let $\varepsilon\to 0$, so that $\sin(N\varepsilon)\to N\varepsilon$ and $\sin(\varepsilon)\to\varepsilon$. The final result is therefore

$$I = N^2 E_o^2$$

showing that these positions on the graph do not correspond to minima but to maxima. Note also that as the value of N increases the intensity of these maxima gets large very fast.

2 $\cos(2A) = 1 - 2\sin^2(A)$