Propagation of Errors

One independent variable

Suppose that we have a dependent variable which depends on one independent variable, that is y=y(x). We measure the independent variable to be $x_o \pm \sigma$, where σ is the uncertainty in the measurement. This has to be estimated before we go any further, and is determined by the design of the experiment. Often it is a “best guess”, and experience will enable you to estimate it.

The question is, “Given the uncertainty in x, what is the uncertainty in y?” Assuming that the uncertainty in x is not too large then we shall make the approximation

$$\frac{dy}{dx}=\frac{\delta y}{\delta x}=\frac{\sigma_y}{\sigma_x}$$

then the uncertainly in y is given by

$$\sigma_y=\frac{dy}{dx}\sigma_x$$

where dy/dx is evaluated at x_o.

1. **Example 1**: Given $y = 4x^3 + 5x + 3$, and $x_o = 2 \pm 0.004$, what is y?
   a. $y = 4(2)^3 + 5(2) + 3 = 45$
   b. $dy/dx = 12x^2 + 5 = 12(2)^2 + 5 = 53$
   c. $\sigma_y = 53*0.004 = 0.212$
   d. $y = 53 \pm 0.2$ (after rounding)

2. **Example 2**: Given y = x sin(x), and $x_o = \pi/4 \pm 0.02$, what is y?
   a. $y = \pi/4 \sin(\pi/4) = \pi/4\sqrt{2} = 0.555$
   b. $dy/dx = \sin(x) + x \cos(x) = \sin(\pi/4) + \pi/4 \cos(\pi/4) = 1.26$
   c. $\sigma_y = 1.26* 0.02 = 0.0252$
   d. $y = 0.555 \pm 0.025$ (after rounding)

Note: this will fail if you make measurements close to a maximum or a minimum in y, for which dy/dx is zero or close to zero. These examples have to be considered on a case by case basis.

Two independent variables

Suppose that we have a dependent variable which depends on two independent variables, that is z=z(x,y). We measure the independent variables to be $x_o \pm \sigma_x$ and $y_o \pm \sigma_y$, where σ_x and σ_y are the uncertainties in the measurement.

The question is, "Given the uncertainties in x and y, what is the uncertainty in z?" We will adapt the formula above analogous to the use of vectors in three dimensional space

$$\sigma_z = \sqrt{\left[\frac{dz}{dx}\right]^2 \sigma_x^2 + \left[\frac{dz}{dy}\right]^2 \sigma_y^2}$$

3. **Example 3:** Given $z = x^2 + y$, $x_o = 2 \pm 0.04$, and $y_o = 4 \pm 0.5$, what is z?
   a. $z = 2^2 + 4 = 8$
   b. $dz/dx = 2x = 4$
   c. $dz/dy = 1$
   d. $\sigma_z = \{8^2 0.04^2 + 1^2 0.5^2\}^{½} = 0.59$
   e. $y = 8 \pm 0.6$ (after rounding)

4. **Example 4:** Given $z = x^2y^4$, $x_o = 6 \pm 0.4$, and $y_o = 1 \pm 0.05$, what is z?
   a. $z = 6^2\ 1^4 = 36$
   b. $dz/dx = 2xy^4 = 12$
   c. $dz/dy = 4x^2y^3 = 144$
   d. $\sigma_z = \{12^2 0.4^2 + 144^2 0.05^2\}^{½} = 8.65$
   e. $y = 36 \pm 8.7$ (after rounding)

More than two independent variables

Simply take the last equation above and add more terms. For example if z = z(xy,u,v) then

$$\sigma_z = \sqrt{\left[\frac{dz}{dx}\right]^2 \sigma_x^2 + \left[\frac{dz}{dy}\right]^2 \sigma_y^2 + \left[\frac{dz}{du}\right]^2 \sigma_u^2 + \left[\frac{dz}{dv}\right]^2 \sigma_v^2}$$