Series

Introduction

The use of series can be a powerful method for solving physics problems under a variety of circumstances

- Problems which have no easy analytical solution, but for which an approximate solution can be found. An example is the electrostatic field produced by a pair of equal but opposite charges (a dipole) in the approximation that the distance the dipole is large in comparison to the dimensions of the dipole.
- Problems for which an analytic solution can be found, but for which the solution is difficult to use. An example is the Langevin function which is encountered in a treatment of the polarization (magnetization) produced by reorientation of existing dipole by an electric (magnetic) field.

Definition of a series

A general description of a series is

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\dots\dots\dots = \sum_{i=0}^{\infty} a_i x^i \qquad (1)$$

We need a method to find the coefficients a_i. If we take this description and differentiate n times, then

$$\frac{d^n f}{dx^n} = [n(n-1)(n-2)...1]\, a_n + [(n+1)n(n-1)...2]\, a_{n+1} x + [(n+2)(n+2)n...3]\, a_{n+2} x^2 + \dots\dots$$
$$= \sum_{i=n}^{\infty} \frac{i!}{(i-n)!} a_i x^{i-n} \qquad (2)$$

Finally if we put x=0 into equation (2) the right hand side becomes a single term, that with i=n. (Note 0!=1.) Solving for a_n we have the result known as the McLaurin series

$$f(x) = f(0) + \frac{df}{dx}(0)\, x + \frac{1}{2!}\frac{d^2 f}{dx^2}(0)\, x^2 + \frac{1}{3!}\frac{d^3 f}{dx^3}(0)\, x^3 + \dots\dots\dots\dots = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n f}{dx^n}(0)\, x^n$$

The Taylor series is similar, except that instead of being useful for small values of x close to zero, it is used for values of x close to some fixed value a. With x=a+h

$$f(a+h) = f(a) + \frac{df}{dx}(a)\, h + \frac{1}{2!}\frac{d^2 f}{dx^2}(a)\, h^2 + \frac{1}{3!}\frac{d^3 f}{dx^3}(a)\, h^3 + \dots\dots\dots\dots = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n f}{dx^n}(a)\, h^n$$

Binomial Series

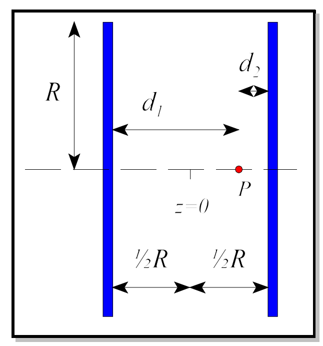
If we set $f(x)=(1+x)^n$, and evaluate the derivatives in the McLaurin series, then we get the result

$$
\begin{aligned}
(1+x)^n &= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots \\
&= \sum_{i=0}^{\infty} \frac{n!}{i!(n-i)!} x^i
\end{aligned}
\qquad \textbf{(5)}
$$

Note that if n is a positive integer this series terminates with the term with i=n. For n a negative integer, or if n is non-integral, the series never terminates.

A physical example - Helmholtz coils

The arrangement known as the Helmholz coils consists of the parallel coils each of N turns and radius R, separated by a distance R. It has the property, as we shall show that the field along the axis close to the center, is nearly uniform.



Helmholtz coils

The field due to a single coil of N turns and radius R, at a distance d along the axis is given by

$$
B = \frac{\mu_o}{2\pi} \frac{m}{(d^2 + R^2)^{3/2}} \qquad \textbf{(6)}
$$

where $m = NIA = Ni\pi R^2$ is the magnetic moment of the coil. For the Helmholtz coils we have two such terms. At the center of the arrangement (the origin for the axes) d=½R for each and so

$$
B = \frac{\mu_o}{2\pi} \frac{16}{5\sqrt{5}} \frac{m}{R^3} \qquad \textbf{(7)}
$$

To calculate the field at a point P along the axis close to the center, let z be the position along the axis relative to the center, in which case d_1=½R+z for one coil and d_2=½R-z for the other. The total field is therefore

$$
\begin{aligned}
B &= \frac{\mu_o}{2\pi} \frac{m}{((\tfrac{1}{2}R+z)^2 + R^2)^{3/2}} + \frac{\mu_o}{2\pi} \frac{m}{((\tfrac{1}{2}R-z)^2 + R^2)^{3/2}} \\
&= \frac{\mu_o}{2\pi} \frac{8}{5\sqrt{5}} \frac{m}{R^3} \left[\frac{1}{\left(1+\frac{4z}{5R}+\frac{4z^2}{5R^2}\right)^{3/2}} + \frac{1}{\left(1-\frac{4z}{5R}+\frac{4z^2}{5R^2}\right)^{3/2}} \right]
\end{aligned}
\qquad \textbf{(8)}
$$

Each bracket can be expanded as a Binomial Series, setting n=-3/2 and

$$x = \pm \frac{4z}{5R} + \frac{4z^2}{5R^2} \qquad \textbf{(9)}$$

with the '+' sign in the first term, and the '-' sign in the second. The computation is tedious. Each series has to be computed to at least terms in x^4, since the lower order terms cancel. The final result is

$$B = \frac{\mu_o}{2\pi}\,\frac{16}{5\sqrt{5}}\,\frac{m}{R^3}\left[1 - \frac{144}{125}\frac{z^4}{r^4} + \ldots\right] \qquad \textbf{(10)}$$

The first term corresponds to the previous result for the field at the center (z=0). The second term is then the correction for points close to but not at the center. As can be seen this correction varies as $(z/R)^4$, and since close to the center $z \ll R$ this term is very small, and so B is almost independent of z.

Some common series

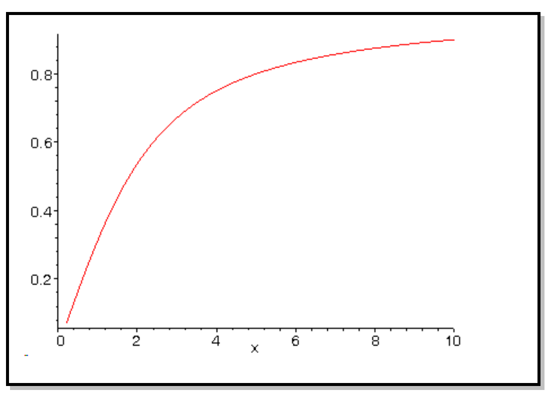
By evaluating the derivatives in the Taylor or McLaurin series an series expression for any f(x) can be found. Some common examples are given below.

$$\begin{aligned}
\sin(x) &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \ldots = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}x^{2n+1} \\
\cos(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \ldots = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}x^{2n} \\
e^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \ldots = \sum_{n=0}^{\infty}\frac{1}{n!}x^n
\end{aligned} \qquad \textbf{(11)}$$

A physical example - the Langevin function

The Langevin function is found in the description of the polarization of a material (such as water) which is composed of molecules which already possess an existing electric dipole moment, and in the magnetization of paramagnetic materials. It has the form

$$L(\lambda) = \coth(\lambda) - \frac{1}{\lambda} = \frac{e^{\lambda}+e^{-\lambda}}{e^{\lambda}-e^{-\lambda}} - \frac{1}{\lambda} \qquad \textbf{(12)}$$



The Langevin function

where λ is defined by the properties of the material and the applied field. It is important to recognize that λ is a very small number for nearly all realistic physical situations. By using a series approximation to this function, it behavior for small values of λ is more easily understood.

We will start by making substitutions for the exponential functions in equation (12) using the expressions in equation (11)

$$L(\lambda) = \frac{(1 + \lambda + \frac{1}{2}\lambda^2 + \frac{1}{6}\lambda^3 + \frac{1}{24}\lambda^4 + ...) + (1 - \lambda + \frac{1}{2}\lambda^2 - \frac{1}{6}\lambda^3 + \frac{1}{24}\lambda^4 + ...)}{(1 + \lambda + \frac{1}{2}\lambda^2 + \frac{1}{6}\lambda^3 + \frac{1}{24}\lambda^4 + ...) - (1 - \lambda + \frac{1}{2}\lambda^2 - \frac{1}{6}\lambda^3 + \frac{1}{24}\lambda^4 + ...)} - \frac{1}{\lambda}$$

$$= \frac{1 + \frac{1}{2}\lambda^2 + \frac{1}{24}\lambda^4 + \frac{1}{720}\lambda^6 + ...}{\lambda + \frac{1}{6}\lambda^3 + \frac{1}{120}\lambda^5 + \frac{1}{5040}\lambda^7 + ...} - \frac{1}{\lambda} \quad \textbf{(13)}$$

The two terms can now be combined over a common denominator

$$L(\lambda) = \frac{\frac{1}{3}\lambda^3 + \frac{1}{30}\lambda^5 + \frac{1}{840}\lambda^7 + \frac{1}{45360}\lambda^9 + ...}{\lambda^2 + \frac{1}{6}\lambda^4 + \frac{1}{120}\lambda^6 + \frac{1}{5040}\lambda^8 + ...} \quad \textbf{(14)}$$

Finally, if we remember that λ is a very small number, then the leading term in each of the numerator and denominator is much larger than any of the succeeding terms. We can therefore drop the succeeding terms, and then cancel factor of λ^2 to get the final result

$$L(\lambda) \approx \frac{1}{3}\lambda \quad \textbf{(15)}$$

which is a much easier expression to understand than that given by equation (12).