2^{nd} Order Differential Equations with Constant Coefficients

Definition

A 2^{nd} order differential equation with constant coefficients has the form

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_o^2 x = f(t) \tag{1}$$

Its solution is comprised of two parts, a complementary function (CF) and a particular integral (PI). The CF is the solution of the equation

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_o^2 x = 0 \tag{2}$$

It will always contain two arbitrary constants, corresponding to the highest power of d/dt in the equation. The PI is *any* solution for x(t) which satisfies equation (1). It contains no arbitrary constants, and only one PI need be found. (There is a Uniqueness Theorem which shows that once one solution has been found, then all solutions have been found.)

Complementary Function

The CF is found be assuming a solution of the form $x = e^{mt}$, where is m is a constant. Substituting into equation (2) we have

$$m^2 x + 2\gamma m x + \omega_o^2 x = 0 \tag{3}$$

After canceling x(t) throughout, equation (3) yields a quadratic equation in m, whose roots are

$$\begin{aligned} m_1 &= -\gamma + \sqrt{\gamma^2 - \omega_o^2} \\ m_2 &= -\gamma - \sqrt{\gamma^2 - \omega_o^2} \end{aligned} \tag{4}$$

This gives two solutions for y(x), and the CF is the linear combination of both, with arbitrary coefficients, A and B

$$x = A e^{m_1 t} + B e^{m_2 t} \tag{5}$$

Case 1

Suppose that γ is small, such that $\gamma^2 < \omega_o^2$. We can write $m = -\gamma \pm i\omega$ where $\omega^2 = \omega_o^2 - \gamma^2$. The complementary function can then be written as

$$
\begin{aligned}
x &= e^{-\gamma t}\left\{A\, e^{i\omega t} + B\, e^{i\omega t}\right\} \\
&= e^{-\gamma t}\left\{A^{/}\sin(\omega t) + B^{/}\cos(\omega t)\right\}
\end{aligned}
\qquad \textbf{(6)}
$$

Physically, in either expression the terms in the parentheses represent an oscillation at frequency ω. The arbitrary coefficients {A,B}, or equivalently {A',B'}, are determined by the initial conditions imposed on y, that is on the value of y and its first derivative at x=0. The exponential in front of the parentheses forces the amplitude of the oscillations to decrease with time, a phenomenon known as damping. The second tern in equation (1) is often referred to as the damping term, and γ as the damping constant.

Case 2

The opposite case, when γ is very large has no oscillatory term. If we write $\alpha^2 = \gamma^2 - \omega_o^2$ then $m_1 = \gamma + \alpha$ and $m_2 = \gamma - \alpha$, both of which are real. The complementary function is therefore

$$
x = A\, e^{-(\gamma-\alpha)t} + B\, e^{-(\gamma+\alpha)t} \qquad \textbf{(7)}
$$

These are both decreasing exponentials which never cross the t axis.

Case 3

This method presents a difficulty in the case where $\gamma^2 = \omega_o^2$, since this leads to the conclusion that $m_1 = m_2$ $(=-\gamma)$. In that case equation (5) takes the form

$$
x = Ae^{-\gamma t} + Be^{-\gamma t} = (A+B)e^{-\gamma t} = Ce^{-\gamma t}
$$

The solution has therefore only one term, with one arbitrary constant, whereas a second order equation always requires two arbitrary constants. This problem is solved by assuming a solution of the form $x = u(t)e^{-\gamma t}$, where u(t) is a function to be determined. Substituting into equation (2) gives

$$
e^{-\gamma t}\frac{d^2u}{dt^2} = 0 \qquad \textbf{(8)}
$$

After canceling the term in $e^{-\gamma t}$, equation (8) can be solved by direct integration to give u = At+B, where A and B are constants. The final solution is therefore

$$
x = (At+B)e^{-\gamma t}
$$

which contains two arbitrary constants, as required of a second order differential equation.

A physical example

A 20 Ω resistor, a 100 nF capacitor, and a 1 mH inductor are connected in series. At t=0 the combination is connected to a 9 V battery. Find the current for t>0.

For elements in series the current is the same, and the voltages add together. We can therefore write

$$L\frac{di}{dt} + iR + \frac{q}{C} = V_o \qquad \textbf{(9)}$$

Differentiating once, and substituting i=dq/dt

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0$$

This is the same as equation (2) if we set

- $\gamma = ½R/L = 10^4$
- $\omega_o = (LC)^{-½} = 10^5$

Since $\gamma<\omega_o$ this corresponds to case 1 above. The circuit will therefore oscillate at a frequency given by

$$\omega^2 = \omega_o^2 - \gamma^2 = (10^5)^2 - (10^4)^2 = 9.9 \text{ x } 10^9$$

and so ω=9.95 x 10^4 rad/s. The oscillation decays with an damping constant $\gamma=10^4$ s^{-1}.

The solution given by equation (6) contains two arbitrary coefficients which are found from the initial conditions. At t=0, when the battery is first connected, the charge on the capacitor is zero, as is the current through the combination. We can therefore write

$$i_{t=0} = e^{-\gamma t}\{A'\sin(\omega t) + B'\cos(\omega t)\}_{t=0} = B' = 0$$

At t=0 all of the applied voltage appears across the inductor, and so

$$V_o = L\,(di/dt)_{t=0} = LA'\ \{\omega e^{-\gamma t}\cos(\omega t) - \gamma e^{-\gamma t}\sin(\omega t)\}_{t=0} = \omega LA'$$

The final solution is therefore

$$i = \frac{V_o}{\omega L}e^{-\gamma t}\sin(\omega t) = .0905\,e^{-10^4 t}\sin(9.95x10^4 t)$$

Particular Integral

Let us return to the original equation

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_o^2 x = f(t) \qquad \textbf{(12)}$$

Before the complementary function can be found the function f(t) must be defined. In principle f(t) can be any function of time. We shall choose the sinusoidal function as the most common in physics problems. We could write $f(t)=A\cos(\omega t)$, although it is easier to use the complex notation $f(t)=Ae^{i\omega t}$ and remember that we only want the real part.

Since the driving function f(t) is sinusoidal with time it is reasonable to assume that x will also be sinusoidal with time, that is we will write $x(t)=x_o e^{i\omega t}$. Here x_o is the amplitude of the oscillation, and in general is itself complex, giving us both a real amplitude and a phase of the oscillation relative to the driving function.

Substituting our expressions for x(t) and f(t) into equation (12) we get

$$-\omega^2 x_o e^{i\omega t} + 2i\gamma\omega\, x_o e^{i\omega t} + \omega_o^2\, x_o e^{i\omega t} = Ae^{i\omega t}$$

and so

$$\begin{aligned} x(t) = x_o e^{i\omega t} &= \frac{A}{(\omega_o^2-\omega^2)+2i\gamma\omega}\, e^{i\omega t} \\ &= \frac{A}{\sqrt{(\omega_o^2-\omega^2)^2+4\gamma^2\omega^2}}\, e^{i\omega t+\varphi} \\ &= \frac{A}{\sqrt{(\omega_o^2-\omega^2)^2+4\gamma^2\omega^2}}\cos(\omega t+\varphi) \end{aligned} \qquad \textbf{(13)}$$

where $\tan\varphi = 2\gamma\omega/(\omega_o^2-\omega^2)$, and the last line recognizes that we only need the real part of the result.

Let us look at the amplitude of the oscillation as a function of frequency. As $\omega\to 0$ or $\omega\to\infty$ the denominator becomes much larger than the numerator, and the amplitude of the oscillation is small. However, when $\omega=\omega_o$ the first term in the denominator vanishes, making the amplitude a maximum, equal to $A/2i\gamma\omega$. This condition is known as resonance. It can be very large, particularly when the damping coefficient γ is small. The resonance curve is a plot of the amplitude as a function of frequency. It typically has the form of a bell curve. The figure to the right shows the resonance curve for the RLC series combination above.