Partial Differential Equations

Introduction

Partial differential equations are those which involve derivatives with respect to more than one variable. Some examples are:

- The Wave Equation
- Diffusion Equation
- The Thermal Conductivity Equation (Heat Diffusion Equation)
- Schrodinger's Equation (Quantum Mechanics)
- Laplace's Equation

The simplest method of solving these equations is by separating the differential equation into two (or more) ordinary differential equations, which can then be solved separately, and their solutions combined to give that for the original partial differential equation.

Thermal Conductivity Equation in One Dimension

Consider a problem in which heat flows in only one direction, for example heat flowing along a solid bar on uniform cross section, assuming no heat loss from the sides. The temperature of the bar is a function of position along the bar, and time (at least until a steady state solution is found). Take a section of the bar of length dx. Heat flows into this section at coordinate x, and leaves at coordinate x+dx. If these are not equal then the difference must result in a change in the temperature of this section of the bar. We can then write

$$Q_x - Q_{x+dx} = -KA\left[\frac{\partial T}{\partial t}\right]_x + KA\left[\frac{\partial T}{\partial t}\right]_{x+dx} = dm\, c\,\frac{\partial T}{\partial t} \qquad \textbf{(1)}$$

where K is the thermal conductivity of the material, c is its specific heat, and dm is the amount of mass with the length dx. By expanding the last term as a Taylor series in dx, and dropping terms in $(dx)^2$ and above we get the Thermal Conductivity Equation

$$\frac{\partial^2 T}{dx^2} = \frac{1}{k}\frac{\partial T}{\partial t} \qquad \textbf{(2)}$$

where we have substituted dm = ρdv = ρAdx, and k =K/cρ.

Solving the Thermal Conductivity Equation

We will look for a solution to the Thermal Conductivity Equation in which the temperature depends on position and time as the product of two separate functions

$$T(x,t) = X(x)\, f(t)$$

Substituting into equation (2) and rearranging we get

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{k}\frac{1}{f}\frac{df}{dt} \tag{3}$$

The left hand side of this equation is only a function of x, and must be time independent, which means that the right hand side must be time independent also. However the right hand side only contains the position independent function f(t). Since we have concluded that it is independent of both position and time, then it must be a constant. If we call this constant $-\lambda^2$ then we can split the partial differential equation (3) into two ordinary differential equations

$$\begin{aligned} \frac{\partial^2 X}{dx^2} &= -\lambda^2 X \\ \frac{df}{dt} &= -\lambda^2 k f \end{aligned} \tag{4}$$

These equations can be solved directly, to give the solution to the partial differential equation

$$T(x,t) = X(x)f(t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-k\lambda^2 t} \tag{5}$$

The arbitrary constants A and B are found from the initial conditions. We are now left with the question of the value of the constant λ which was introduced during the process of separating variables. In principle any value of λ is acceptable, and so from the Principle of Superposition the final solution is the sum of terms given by equation (5) with all possible λ

$$T(x,t) = \sum_{n=1}^{\infty} (A_n\cos\lambda_n x + B_n\sin\lambda_n x)\, e^{-k\lambda_n^2 t} \tag{6}$$

Using the Initial Conditions

Having found a general solution to the one dimensional heat conduction equation, let is now further specify our problem. If a bar of length L is made from two identical pieces of length L/2 which are initially held at temperatures T=0 and $T=T_o$ respectively. At t=0 these two pieces are placed in contact so that heat can flow from the hotter section to the cooler section. Setting the x coordinate origin at the junction of the two sections, at t=0 we can write

$$\begin{aligned} T(x,0) = \sum_{n=1}^{\infty} (A_n\cos\lambda_n x + B_n\sin\lambda_n x) &= 0 \qquad -\tfrac{1}{2}L<x<0 \\ &= T_o \qquad 0<x<\tfrac{1}{2}L \end{aligned} \tag{7}$$

The constants can now be found by treating equation (7) as a Fourier Series in x. The result is