Fourier Series

Introduction

We need to express a function f(x), which is defined between x=0 and x=L, as a series of sine and cosine functions

$$f(x) = a_o + \Sigma\, a_n \cos(2n\pi x/L) + \Sigma\, b_n \sin(2n\pi x/L)$$

where each sum is from n=1 to n=infinity. The trick is to find the values of the coefficients a_o, a_n, and b_n.

Note: We can always use this series providing either

- The function f(x) is periodic with a spacing of L.
- The function f(x) is only defined within the range x=0 to x=L, and we do not care what its value is outside this range. (The Fourier series will make it periodic if you try to calculate or plot f(x) outside the defined range.)
- All integrals have limits of x=0 to x=L. (It is difficult to write this using the word processor.)

Theory

Since the right hand side of the equation contains two infinite sums the task of finding the individual coefficients is not straightforward. We need a 'filter' to extract the individual terms. This is achieved by multiplying both sides of the equation by another sinusoidal function, and then integrating from x=0 to x=L. For example, if we multiply by cos(2mπx/L) where m is any integer, and integrate

$$\begin{aligned}\int f(x) \cos(2m\pi x/L)\, dx = {} & a_o \int \cos(2m\pi x/L)\, dx \\ & + \Sigma\, a_n \int \cos(2n\pi x/L) \cos(2m\pi x/L)\, dx \\ & + \Sigma\, b_n \int \sin(2n\pi x/L) \cos(2m\pi x/L)\, dx\end{aligned}$$

Each of these integrals has to be evaluated individually. The results are

- $\int \cos(2m\pi x/L)\, dx = 0$ for all values of m
- $\int \sin(2n\pi x/L) \cos(2m\pi x/L)\, dx = 0$ for all values of m
- $\int \cos(2n\pi x/L) \cos(2m\pi x/L)\, dx = 0$ for all values of m except m=n
- $\int \cos(2n\pi x/L) \cos(2m\pi x/L)\, dx = L/2$ if m=n

The equation above contains 2(infinity) +1 integrals, but all are zero except one (the cosine term with n=m) and so

$$\int f(x) \cos(2m\pi x/L)\, dx = a_m L/2$$

and so

$$a_m = 2/L \int f(x) \cos(2m\pi x/L)\, dx$$

Similarly if we multiply by $\sin(2m\pi x/L)$ where m is any integer, and integrate we can show that

$$b_m = 2/L \int f(x) \sin(2m\pi x/L)\, dx$$

The last term to find is a_o. Without multiplying by any sinusoidal function, just integrate

$$\int f(x)\, dx = a_o \int dx + \Sigma\, a_n \int \cos(2n\pi x/L)\, dx + \Sigma\, b_n \int \sin(2n\pi x/L)\, dx$$

Each of the sine and cosine integrals on the right hand side is zero, and so

$$\int f(x)\, dx = a_o \int dx = a_o L$$

and so

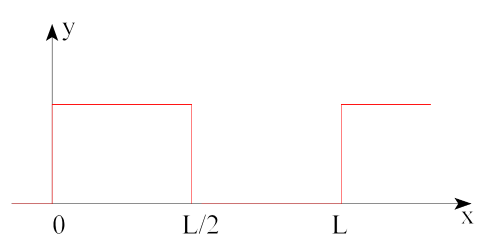
$$a_o = 1/L \int f(x)\, dx$$

The three boxed equations allow us to calculate all the coefficients for the Fourier Series. (Note the absence of the factor of 2 at the front of the equation for a_o.)

Example 1 - The Square Wave

The square wave is defined by

- $f(x) = 1$ for $0<x<L/2$
- $f(x) = 0$ for $L/2<x<L$



The integrals in the boxes (strictly speaking from x=0 to x=L) have new limits from x=0 to x=L/2. All contributions to the integrals for x>L/2 are zero. We have then

- $a_o = 1/L \int dx = 1/L\, (L/2) = ½$
- $a_n = 2/L \int \cos(2n\pi x/L)\, dx = 2/L\, (L/2n\pi)\, [\sin(2n\pi x/L)] = 0$, since the sin is zero for both x=0 and x=L/2
- $b_n = 2/L \int \sin(2n\pi x/L)\, dx = 2/L\, (-L/2n\pi)\, [\cos(2n\pi x/L)] = (1/n\pi)\, \{1-\cos(n\pi)\}$. The value of this depends on n. If n is even $\cos(n\pi)=1$ and so $b_n=0$. However if n is odd $\cos(n\pi)=-1$ and so $b_n=2/n\pi$.

Putting this all together, and substituting the values that we have calculated for the coefficients into the Fourier Series equation at the top of the first page

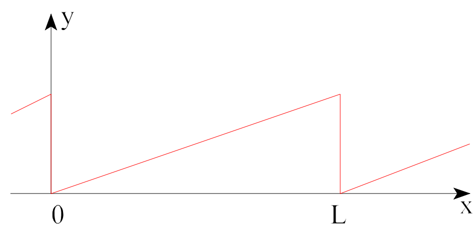
$$f(x) = ½ + \Sigma\ (2/n\pi)\ \sin(2n\pi x/L)$$

where the sum is over the odd terms (odd values of n) only

$$f(x) = ½ + (2/\pi)\ \{\sin(2\pi x/L) + (1/3)\ \sin(6\pi x/L) + (1/5)\ \sin(10\pi x/L) + ...\ \}$$

Example 2 - The Sawtooth Wave

The sawtooth wave is defined by f(x) = x/L for all values of x in the range x=0 to x=L. We have then



- $a_o = 1/L \int x/L\ dx = 1/L^2\ (L^2/2) = ½$
- $a_n = 2/L \int x/L\ \cos(2n\pi x/L)\ dx = 0$. (You will need to integrate by parts to get this result.)
- $b_n = 2/L \int x/L\ \sin(2n\pi x/L)\ dx = (-1/n\pi)$ (Again, integrate by parts, see below.)

The net result is therefore

$$f(x) = ½ + \Sigma\ (-1/n\pi)\ \sin(2n\pi x/L)$$

or

$$f(x) = ½ - (1/\pi)\ \{\sin(2\pi x/L) +1/2\ \sin(4\pi x/L) + 1/3\ \sin(6\pi x/L) + ...\ \}$$

Appendix - Integrating by parts

We will use the method

$$\int u\ dv = uv - \int v\ du$$

to find $\int x \sin(ax)\ dx$ between x=0 and x=L, where $a = 2n\pi/L$. Let

- u = x
- dv = sin(ax) dx, in which case v = (-1/a) cos(ax) by direct integration.

Then

$$\int x \sin(ax)\ dx = (-x/a) \cos(ax) - \int (-1/a) \cos(ax)\ dx$$
$$= (-x/a) \cos(ax) + (1/a^2) \sin(ax)$$

Putting in $a = 2n\pi/L$ then sin(ax)=0 at both limits, and (x/a)cos(ax)=0cos(0)=0 at the bottom limit, and $(L/(2n\pi/L)) \cos(2n\pi) = L^2/2n\pi$ at the top limit. We have then the result

$$\int x \sin(2n\pi x/L)\ dx = L^2/2n\pi$$

and the result for b_n above follows.

Variations

Some Fourier series problems use a slightly different interval. There are three common variations. For each the basic method is the same.

- If the interval is from x=0 to x=1, then replace L with 1 in every equation, and integrate from 0 to 1.

$$f(x) = a_o + \Sigma\, a_n \cos(2n\pi x) + \Sigma\, b_n \sin(2n\pi x)$$

- If the interval is from x=0 to x=2L, then replace L with 2L in every equation, and integrate from 0 to 2L.

$$f(x) = a_o + \Sigma\, a_n \cos(n\pi x/L) + \Sigma\, b_n \sin(n\pi x/L)$$

- If the interval is from x=-L to x=L, then replace L with 2L in every equation, and integrate from -L to +L.

$$f(x) = a_o + \Sigma\, a_n \cos(n\pi x/L) + \Sigma\, b_n \sin(n\pi x/L)$$

Fourier series using other basis functions

In principle any orthogonal set of functions can be used for a Fourier series. For example the Legendre polynomials satisfy the conditions

$$\int_0^\pi P_l(\theta)\, P_m(\theta) \sin\theta\, d\theta = \int_{-1}^{+1} P_l(x)\, P_m(x)\, dx = 0 \qquad \textbf{(1)}$$

and

$$\int_0^\pi P_l^2(\theta) \sin\theta\, d\theta = \int_{-1}^{+1} P_l^2(x)\, dx = \frac{2}{2l+1} \qquad \textbf{(2)}$$