Complex Numbers

Definition

The square of any real number is always positive, there is no real number whose square is negative. In order to deal with the square root of a negative number we introduce the symbol 'i' to mean √(-1). Using this symbol we can then write an expression for the square root of a negative number

- √(-4) = √(-1)√4 = ±2 i
- √(-6) = √(-1)√6 = ±√6 i
- and in general, √(-a) = √(-1)√a = ±√a i

Numbers which have this form are known as *imaginary* numbers. If we add together a real number and an imaginary number then we get a *complex* number. Examples are

- 2 + 3i
- a + ib
- x + iy

When using Maple Re(z) finds the real part of the complex number z, and Im(z) finds the imaginary part.

```
>Re(4+3*I);
                4
> Im(4+3*I);
                3
```

Complex arithmetic

Complex arithmetic follows the same rules as arithmetic with real numbers.

- Addition
  - (2+3i) + (4-i) = (2+4) + (3-1)i = 6 + 2i
- Subtraction
  - (2+3i) - (4-i) = (2-4) + (3+1)i = -2 + 4i
- Multiplication
  - (2+3i) * (4-i) = 2*4 + 2*(-i) + (3i)*4 +(3i)*(-i)
    = 8 -2i +12i -3i²
    = 8 -2i +12i +3   (Remember i²=-1)
    =11 + 10i
- Division - see below

Complex conjugate

The complex conjugate of a complex number is the number whose real part is unchanged, and whose imaginary part has the opposite sign.

- The complex conjugate of 2+3i is 2-3i
- The complex conjugate of 4-i is 4+i
- The complex conjugate of x+yi is x-yi

To find the complex conjugate of z, use the function conjugate(z)

```
>conjugate(4+7*I);
                4 - 7 I
```

In conventional notation the complex conjugate of z is written as z^*.

Magnitude of a complex number

The complex conjugate is a useful concept when using complex numbers in physics because the product of any complex number and its own complex conjugate is always a real number

$$zz^* = (x+iy) * (x-iy) = x^2 + x*(-iy) + (iy)*x + (iy)*(-iy) = x^2 + y^2$$

This is used to define the magnitude of a complex number, written as |a| and defined by

$$|z| = \sqrt{(zz^*)}$$

In Maple to find the magnitude of a complex number use the abs(z) function

```
> abs(4+7*I);
                    √65
```

Inverse of a complex number

Taking the inverse of a complex number is slightly more complicated. It can be evaluated by multiplying the expression by the complex conjugate of the denominator and using the previous relationship

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy}\,\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{z^*}{|z|^2}$$

For example, if a=1+2i, then $|a|^2=1^2+2^2 = 5$, and $a^{-1} = (1 - 2i)/5$.

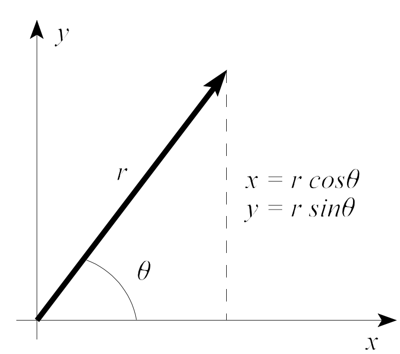
Division

As with arithmetic of real numbers, complex division is equivalent to multiplying the numerator by the inverse of the denominator. For example

$$\frac{4+12i}{1+2i} = 4+12i * \frac{1-2i}{5} = \frac{28}{5} + \frac{4}{5}i$$

Polar notation

Although the {x,y} notation is intuitive for defining the complex number x+iy mathematical operations using this notation can become cumbersome very quickly. For example, try computing the result of $(2+i)^6$. Most often it is more convenient to use a polar notation using the standard polar relationships between {x,y} and {r,θ} (See diagram to right.) The complex number can the be written as



$$x+iy = r\cos\theta + r\sin\theta\, i = r(\cos\theta + i\sin\theta)$$

The parentheses in the last expression are then used to define a complex exponential $e^{i\theta} = \cos\theta + i\sin\theta$ and the complex number can the also be written as $re^{i\theta}$. The two forms are equivalent, and conversion from one to the other is straightforward. When defined this way the complex exponential obeys the same operational logic that real exponentials do.

Note

- The conversion from $x+iy$ to $re^{i\theta}$ is not unique. The angle θ is defined by the equation $\tan\theta = y/x$, which has repeating roots. As a result $re^{i\theta}$ can also be written as $re^{i(\theta+2p\pi)}$, where p is any integer.
- The complex conjugate of $e^{i\theta}$ is $e^{-i\theta}$
- $(e^{i\theta})(e^{-i\theta}) = 1$.
- If $z = re^{i\theta}$, then r is the magnitude of z, that is $|z|^2 = zz^* = r^2$.
- $e^{i0} = 1$
- $e^{i\pi} = -1$
- $e^{i\pi/2} = i$
- $e^{i\pi/4} = (1+i)/\sqrt{2}$

In Maple to find r and θ of a complex number use the functions abs(z) and argument(z). The function polar(z) returns both

```
> abs(4+7*I);
                √65
> argument(4+7*I);
               arctan(7/4)
> readlib(polar):polar(4+7*I);
                   polar(√65, arctan(7/4))
```

Arithmetic using complex notation

- Multiplication
  - $re^{i\theta} * Re^{i\Theta} = rRe^{i(\theta+\Theta)}$
- Division
  - $re^{i\theta} / Re^{i\Theta} = (r/R)e^{i(\theta-\Theta)}$
- Inverse
  - $(re^{i\theta})^{-1} = 1/r\ e^{-i\theta}$
- Powers
  - $(re^{i\theta})^n = r^n\ e^{in\theta}$

Complex roots of numbers

The equation $z^n = a^n$ should have n roots each of them different. With real numbers this is not true. For example the equation $x^3=8$ has only one real root, $x=2$. This is remedied using complex numbers by recognizing that 8 can also be written as $8e^{i2p\pi}$, where p is any integer. If we try to solve the equation $z^3=8$ then

$$z^3 = 8\ e^{i2p\pi} = 2^3\ e^{i2p\pi}$$

and so

$$z = (2^3\ e^{i2p\pi})^{1/3} = 2\ e^{i2p\pi/3}$$

In principle any integer p is valid, although substituting p=3 give the same result as substituting p=0, substituting p=4 give the same result as substituting p=1, and so on. The only unique results correspond to p=0,1, or 2. The three (distinct) roots of 8 are therefore

- when p=0, $z = 2e^{i0} = 2$
- when p=1, $z = 2e^{i2\pi/3} = -1 + \sqrt{3}\, i$
- when p=2, $z = 2e^{i4\pi/3} = -1 - \sqrt{3}\, i$

Oscillators, waves, and complex numbers

Review of oscillator physics

A simple harmonic oscillator is one which satisfies the differential equation

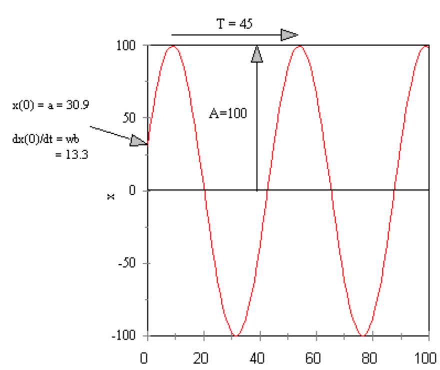
$$\frac{d^2x}{dt^2} = -\omega^2 x \qquad \textbf{(3)}$$

in which ω is a (real) constant which depends on the properties of the oscillator. Oscillators are found in almost all areas of physics, including

- A mass (m) on a spring of spring constant k, for which $\omega^2 = k/m$.
- An electrical oscillator formed by a capacitor (C) and an inductor (L), for which $\omega^2 = 1/LC$.
- A simple pendulum, consisting of a mass (m) at the end of a string of length L. In the small angle limit this is a simple harmonic oscillator with $\omega^2 = g/L$.

The solution of equation (5) can take one or other of the equivalent forms

- $x = a \cos(\omega t) - b \sin(\omega t)$, where a and b are constants to be determined from initial conditions.
- $x = A \cos(\omega t + \varphi)$, where A and φ are the constants to be determined from initial conditions. In this form A is known as the *amplitude*, being the maximum possible value of x, and φ is called the *phase angle*.



Either form can be readily converted to the other, providing we make the associations $a=A\cos\varphi$ and $b=A\sin\varphi$. If we substitute t=0 then it can be seen that a is also the value of x at t=0, and after differentiating once, that the slope (dx/dt) at t=0 is equal to $-\omega b$.

The solution using either of these expressions is periodic. Starting from any point on the curve, after a one cycle the curve returns to the same value of x. The time it takes for one cycle is known as the *time period* (T). Since the trigonometric functions repeat every 2π radians, it follows that $\omega=2\pi/T$. From the time period we can also define the frequency of the oscillator (f) given by $f = 1/T = \omega/2\pi$. (Note: ω is known as the angular frequency.)

A plot of the solution with A=100, T=45 s and $\varphi=-0.4\pi$ is shown in the diagram to the right. For this oscillator $\omega = 2\pi/45 = 0.14$ rad/s and $f = 1/T = 0.022$ Hz.

Oscillators and Energy

All oscillators have at least two energy terms associated with them. For the examples above

- The mass on a spring has the kinetic energy of the mass ($\frac{1}{2}mv^2$) and the stored elastic energy of the spring ($\frac{1}{2}kx^2$).
- An electrical oscillator has the energy stored in the electric field of the capacitor ($\frac{1}{2}CV^2$) and the energy stored in the magnetic field of the inductor ($\frac{1}{2}LI^2$).
- A simple pendulum has the kinetic energy of the mass ($\frac{1}{2}mv^2$) and its gravitational potential energy ($mgh = mgL(1-\cos\theta) \approx \frac{1}{2}mgL\theta^2$).

An oscillator continually exchanges energy between the energy types which are available, although the total energy is a constant. For example, in the case of the mass on the spring, when the spring has its maximum extension the mass is not moving, and all of the energy is in the form of the elastic energy of the spring, $\frac{1}{2}kA^2$. One quarter of a cycle later the spring has returned to its unstretched length, and the mass reaches its maximum speed. All of the energy has been converted to kinetic energy of the mass. In the next quarter of a cycle the energy exchange reverses, until all of the energy is again stored in the spring, after which the process repeats indefinitely.

Using complex notation

Using real notation the solution of equation (5) involves trigonometric functions. Algebraic manipulation of these functions can be cumbersome. A significant reduction in complexity can follow by switching to complex notation, in which case the solution to equation (5) is the real part of a complex expression

$$x = Ae^{\pm i(\omega t+\varphi)}$$

The symbols have the same meaning as before.

For example, suppose we want to add together three different solutions, all of the same amplitude and frequency, but with phases 0, $2\pi/3$, and $4\pi/3$. Using real notation this is

$$x = A\,[\cos(\omega t) + \cos(\omega t+2\pi/3) + \cos(\omega t+4\pi/3)\,]$$

This expression can be simplified (eventually!). However in complex notation the same addition becomes

$$
\begin{aligned}
x &= A\left[e^{i\omega t} + e^{i(\omega t+2\pi/3)} + e^{i(\omega t+4\pi/3)}\right] \\
&= A\,e^{i\omega t}\left[1 + e^{i2\pi/3} + e^{i4\pi/3}\right] \\
&= A\,e^{i\omega t}\left[1 + \left(-\frac{1}{2}+\frac{2i}{\sqrt{3}}\right) + \left(-\frac{1}{2}-\frac{2i}{\sqrt{3}}\right)\right] \\
&= 0
\end{aligned}
$$

A similar simplification arises when finding derivatives or integrals. In real notation the trigonometric functions alternate between cos(ωt+φ) and sin(ωt+φ) each time a derivative is taken. In complex notation the derivative of the exponential $e^{i(\omega t+\varphi)}$ returns the same exponential.

Review of waves

Waves arise as the solution to the Wave Equation, which contains both time and position derivatives

$$
\frac{\partial^2 x}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 x}{\partial t^2} \qquad \textbf{(5)}
$$

The simplest (single frequency) solution to the Wave Equation is

$$
x = A\cos(\kappa z-\omega t+\varphi)
$$

In this equation A, ω, and φ are arbitrary constants with the same meaning that they had for the oscillator, and κ=ω/v.

Since x depends on both z and t it is sinusoidally oscillatory with respect to either variable.

- If z is fixed, that is we look at any given point on the wave, then x oscillates in time with an angular frequency ω. The time period is T=2π/ω and the frequency is f=ω/2π.
- If t is fixed, that is we take a snapshot of the wave, then x varies sinusoidally with z. The constant κ is called the wavenumber, and defines the wavelength λ=2π/κ which is the distance between successive peaks of the oscillation. If we combine this definition with that for κ then we get a pair of equivalent expressions for the speed of the wave

$$
v = \frac{\omega}{\kappa} = \frac{2\pi f}{2\pi/\lambda} = f\lambda \qquad \textbf{(6)}
$$

Displaying a wave: Animate

To produce an animation of a wave in Maple use the animate function. Note it is included in the 'plots' library, which must be called.

```
> with(plots):
```

The next line defines the wavelength and the period

```
> L:=2; T:=1;
```

Define the function covering two wavelengths and one time period, and then run the animation.

```
> f:=cos(2*Pi*z/(2*L)-2*Pi*t/T);
> animate(f,z=0..2*L,t=0..T);
```

Waves and complex notation

The same substitution can be made when dealing with waves, except the solution is now a function of z and t. Instead of writing x = A cos(kz-ωt+φ) it is often preferable to write the solution as the real part of $x = A\,e^{i(kz-\omega t+\varphi)}$.

A word of warning

The energy of an oscillator or of a wave is proportional to its amplitude. For example, the energy transported by an electromagnetic wave (known as the Poynting vector) in a vacuum is given by $S = E^2/\mu_o c$, where E is the amplitude of the electric field. When working with a complex field $\mathscr{E}$ ($=\mathscr{E}_o e^{i\omega t}$) it is tempting to write the Poynting vector as $S = E^2/\mu_o c = |\mathscr{E}|^2/\mu_o c = \mathscr{E}\mathscr{E}^*/\mu_o c$. However, we should remember that the electric field E is equal to the *real* part of $\mathscr{E}$ only. The correct representation in complex notation is

$$S = \frac{E^2}{\mu_o c} = \frac{Re(\mathscr{E})\,Re(\mathscr{E})}{\mu_o c}$$

$$= \frac{\tfrac{1}{2}(\mathscr{E}+\mathscr{E}^*)\,\tfrac{1}{2}(\mathscr{E}+\mathscr{E}^*)}{\mu_o c}$$

$$= \frac{(\mathscr{E}\mathscr{E} + \mathscr{E}\mathscr{E}^* + \mathscr{E}^*\mathscr{E} + \mathscr{E}^*\mathscr{E}^*)}{4\mu_o c}$$

In practice, since the electric field in a light wave oscillates at angular frequencies of the order of 10^{15} rad/s, which is too high to follow, we are only interested in the average rate at which energy is transported. In the expression above, the first and last terms average to zero ($\mathscr{E}\mathscr{E} = \mathscr{E}_o\,e^{i\omega t}\mathscr{E}_o e^{i\omega t} = \mathscr{E}_o{}^2 e^{2i\omega t}$) whereas the second and third are constant ($\mathscr{E}\mathscr{E}^* = \mathscr{E}_o\,e^{i\omega t}\mathscr{E}_o{}^* e^{-i\omega t} = \mathscr{E}_o\,\mathscr{E}_o{}^*$). We can then write

$$S = \frac{(\mathscr{E}_o\mathscr{E}_o^* + \mathscr{E}_o^*\mathscr{E}_o)}{4\mu_o c}$$

$$= \frac{2\,\mathscr{E}_o\mathscr{E}_o^*}{4\mu_o c}$$

$$= \frac{|\mathscr{E}_o|^2}{2\mu_o c}$$