Phys 3010 Mathematical Physics
Assignment 18

This assignment contains some very useful relations which you are likely to encounter in subsequent classes.

Preliminaries

1. Show that
   a. div$\mathbf{r}$ = 3
   b. curl($r^n \hat{\mathbf{r}}$) = 0 for any value of n
   c. grad(r^n) = $nr^{n-1}\hat{\mathbf{r}} = nr^{n-2}\mathbf{r}$
   d. ($\mathbf{a}.\nabla$)$\mathbf{r}$ = $\mathbf{a}$, for any vector $\mathbf{a}$, even if it is not constant.

Product rule questions

The rest of the questions should be answered ***without resorting to any particular coordinate system***. Instead use the results of the first question, plus the product rules on the handout from class.

2. If $\mathbf{r}$ is the position vector, $\mathbf{a}$ is any vector, and $\mathbf{m}$ is any constant vector, find the following:
   a. div $r^n\mathbf{r} = r^n$ div($\mathbf{r}$) + $\mathbf{r}$.grad(r^n) = r^n 3 + $\mathbf{r}.nr^{n-2}\mathbf{r} = r^n$ 3 + $nr^{n-2}r^2$ = (n+3)r^n
   b. div ($\mathbf{m}$x$\mathbf{r}$) = $\mathbf{m}$.curl$\mathbf{r}$ + $\mathbf{r}$.curl$\mathbf{m}$ = $\mathbf{m}$.0 + $\mathbf{r}$.0 = 0
   c. curl ($\mathbf{m}$x$\mathbf{r}$)
      i. = ($\mathbf{r}.\nabla$)$\mathbf{m}$ – ($\mathbf{m}.\nabla$)$\mathbf{r}$ + (div$\mathbf{r}$)$\mathbf{m}$ – (div$\mathbf{m}$)$\mathbf{r}$
      ii. = 0 – $\mathbf{m}$ + (3)$\mathbf{m}$ – 0
      iii. = 2$\mathbf{m}$
   d. grad ($\mathbf{m.r}/r^3$)
      i. grad ($\mathbf{m.r}/r^3$) = ($1/r^3$)grad ($\mathbf{m.r}$) + ($\mathbf{m.r}$).grad($1/r^3$)
      ii. = ($1/r^3$)[$\mathbf{m}$xcurl$\mathbf{r}$+$\mathbf{r}$xcurl$\mathbf{m}$+($\mathbf{m}.\nabla$)$\mathbf{r}$+($\mathbf{r}.\nabla$)$\mathbf{m}$] + ($\mathbf{m.r}$).grad($1/r^3$)
      iii. = ($1/r^3$)[$\mathbf{0}$+$\mathbf{0}$+$\mathbf{m}$+$\mathbf{0}$] + ($\mathbf{m.r}$).[-3$\mathbf{r}/r^5$]
      iv. = ($1/r^5$) {$r^2\mathbf{m}$ – 3($\mathbf{m.r}$)$\mathbf{r}$}
3. If f$\mathbf{a}$ = grad(g), show that $\mathbf{a}$.(curl $\mathbf{a}$) = 0, regardless of the functions f and g, except for f=0. (Hint: since the final expression requires you to know curl $\mathbf{a}$, start by taking the curl of both sides of the original expression.)
   a. Take curl of both sides: curl(f$\mathbf{a}$) = curlgrad(g) = 0 for any g
   b. fcurl$\mathbf{a}$ – $\mathbf{a}$xgradf = 0
   c. Dot with $\mathbf{a}$: f$\mathbf{a}$.curl$\mathbf{a}$ – $\mathbf{a}.\mathbf{a}$xgradf = 0
   d. The last term is 0 since it is the scaler triple product with two vectors the same
   e. f$\mathbf{a}$.curl$\mathbf{a}$ = 0
   f. Assuming f≠0 it can be divided to give the result $\mathbf{a}$.curl$\mathbf{a}$ = 0