Additional problems in multivariate calculus

The following problems all require the use of multiple integrals.

Calculating area

Calculate the area of the following. You might want to plot out the curves first to get an idea of what the shape looks like.

1. The astroid defined by the parametric equations $x = a\cos^3 t$ and $y = a\sin^3 t$, where $0 \le t \le 2\pi$.

2. The lemniscate defined by $r^2 = a^2\cos(2\theta)$, which is only defined for $-\pi/4 \le \theta \le \pi/4$ and $3\pi/4 \le \theta \le 5\pi/4$

3. The cardiod defined by $r = a(1-\cos\theta)$ for $0 \le \theta \le 2\pi$.

4. The curve defined by the parametric equations $x = a\sin(2t)$ and $y = b\cos(t)$, where $0 \le < t \le 2\pi$.

Calculating volumes

Calculate the volumes of the following

5. The vertical cylinder $(x-a)^2 + y^2 = a^2$ between the planes $z = 0$ and $z = 4a-x$.

6. The pyramid defined by the points (a,0,0) (0,a,0) (-a,0,0) (0,-a,0) and (0,0,a).

7. The regular 'pyramid' defined by a sphere of radius R and four line each running from the centre of the sphere and inclined at an angle α to the axis of the pyramid so that the four points where the lines meet the sphere form to corners of a square.

Mechanical Work

When a particle moves under the action of a force then the work done is equal to $\int \mathbf{F} \cdot d\mathbf{r}$. Calculate the work done for the following forces and paths.

8. The force $\mathbf{F} = x^2\mathbf{i} + yz\mathbf{j}$ from the point (1,2,3) to the point (4,5,6) along the straight lines which also go through the intermediary points (1,2,6) and (1,5,6).

9. The force $\mathbf{F} = x^2\mathbf{i} + yz\mathbf{j}$ from the point (1,2,3) to the point (4,5,6) along the straight line between them.

10. The force $\mathbf{F} = 3x^2\mathbf{i}$ along one half of the cardiod $6(1+\cos\varphi)$ for $0 \le \varphi \le \pi$.

11. The force $\mathbf{F} = 1/r^2\, \hat{\mathbf{r}}$ for the helix $x = a\cos(2t)$, $y = a\sin(2t)$, $z=t$ for $0 \le t \le 2\pi$.

12. The force $\mathbf{F} = 3r^2\,\hat{\mathbf{r}}$ along one half of the cardiod $r = 6(1+\cos\varphi)$ for $0 \leq \varphi \leq \pi$.

13. The force $\mathbf{F} = 3r^3\,\hat{\mathbf{r}}$ along one half of the lemniscate $r^2 = a^2\cos 2\varphi$ for $-\pi/4 \leq \varphi \leq \pi/4$.

Moment of Inertia

Calculate the following moments of inertia, defined as $\int R^2\, dm$, where R is the distance from the axis of revolution. The mass element dm can take the forms ρdv (for a volume element), σdA (for an area element), or λdl (for a line element).

14. A flat segment of a uniform disk of radius r, if the segment subtends an angle of θ at the origin. Take the axis of rotation to be normal to the disk and passing through the origin.

15. A flat uniform annulus between the limits $a \leq r \leq b$.

16. A solid sphere of radius a about any axis passing through its centre if the density of the sphere varies as $\rho = \rho_o\, r/a$.

17. A ring of radius a lying in the x-y plane with the axis of rotation normal to the plane of the ring and passing through its centre, if the density varies as $\lambda = \lambda_o \cos^2\varphi$.

18. A ring of radius a lying in the x-y plane with the axis of rotation in the plane of the ring along the y axis, and passing through its centre, if the density varies as $\lambda = \lambda_o\,(1+\cos\varphi)$.

19. A square of side a, lying in the x-y plane with one corner at the origin, about an axis which is the z-axis, if the density varies as $\sigma = \sigma_o\, r/a$.

20. The cone defined by $x^2 + y^2 = a^2z^2$ for $0 \leq z \leq 1$ if the axis of rotation corresponds to the z axis.

21. The cone defined by $x^2 + y^2 = a^2z^2$ for $0 \leq z \leq 1$ if the axis of rotation corresponds to the x axis.

Centre of mass

Calculate the centre of mass of the following. The centre of mass is defined as $\int \mathbf{r}\, dm$, where $\mathbf{r}$ is the location of the mass element dm from the origin. The mass element dm can take the forms ρdv (for a volume element), σdA (for an area element), or λdl (for a line element).

22. A flat segment of a uniform disk of radius r, if the segment subtends an angle of θ at the origin.
23. A ring of radius a lying in the x-y plane with its centre at the origin, if the density varies as $\lambda = \lambda_o \cos^2\varphi$.

24. A ring of radius a lying in the x-y plane with its centre at the origin, if the density varies as $\lambda = \lambda_o (1+\cos\varphi)$.

25. A square of side a, lying in the x-y plane with one corner at the origin, if the density varies as $\sigma = \sigma_o r/a$.

26. The cone defined by $x^2 + y^2 = a^2z^2$ for $0 \leq z \leq 1$.

Electrostatics

The following problems all use Coulomb's Law for the electrostatic field due to a distributed charge,

$$\mathbf{E} = \int \frac{\mathbf{R}}{R^3} dq$$

where **R** is the vector from the charge element dq to the point P where the field is to be calculated, and R is its magnitude. The charge element dq can take any one of the forms ρdv (for a volume element), σdA (for an area element), or λdl (for a line element). Find the electric field

27. At a point P which lies a distance a above the centre of a square ring of side 2a which carries a uniform line charge density λ.

28. At a point P which lies a distance a above the centre of a square plate of side 2a which carries a uniform surface charge density σ.

29. At the origin for a uniform line charge on the arc of a circle of radius a and extending from $\varphi=0$ to $\varphi=\alpha$.

30. At the origin for a non-uniform line charge on the arc of a circle of radius a and extending from $\varphi=0$ to $\varphi=\alpha$ if the charge density varies as $\lambda = \lambda_o \cos^2\varphi$.

31. At the origin for a line charge which extends from x=a (a>0) to $x=+\infty$ if the charge varies as $\lambda = \lambda_o a^2/x^2$.

Electric flux is defined as $\Phi = \int \mathbf{E}.d\mathbf{A}$. Find the electric flux in the following examples.

32. A uniform field $\mathbf{E} = E\mathbf{i}$ over the surface of the hemisphere of radius a and x>0.

33. A field $\mathbf{E} = E_o \exp\{-(x^2+y^2)/a^2\}\,\mathbf{k}$, over the surface of a disk of radius a which is normal to the field.