Phys 3320 Electricity & Magnetism II
Spring 2019
Mid Term - April 17th, 2019

Section 1 – in class portion

1. What conditions must be satisfied for Laplace's Equation to be applicable? Distinguish between electric and magnetic fields.
   a. Fields must be time independent
   b. In the case if the electric field
      i. the free charge density (ρ) must be zero (otherwise the field satisfies Poisson's Equation).
      ii. The material must be LIH so that $\mathbf{D} = \varepsilon_r\varepsilon_o\mathbf{E}$
   c. In the case if the magnetic field
      i. the free charge current (**J**) must be zero.
      ii. The material need no be LIH

2. What is the speed of the pressure wave

$$P = 45e^{i(40000y + 25000z - 900t + 1.4)}$$

   a. k = √(20000² + 25000²) = 47,170
   b. ω = 900
   c. v = ω/k = 0.019 m/s

3. Using Laplace's equation show that in a region with no charges and a central potential (that is V=V(r)) then the only functional form for the field is the one given by Coulomb's Law.

   Hint: In spherical polar coordinates

$$\nabla^2 f = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{df}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{d^2 f}{d\phi^2}$$

4. A plane light wave has an intensity of 400 W/m². The magnetic field intensity is 3 μT. Find the speed of the wave. (Assume $\mu = \mu_o = 4\pi \times 10^{-7}$ H/m).

$$<S> = \frac{E^2}{2\mu_o v} = \frac{(vB)^2}{2\mu_o v} = \frac{vB^2}{2\mu_o}$$

$$v = 2\mu_o\frac{<S>}{B^2} = 1.1\,x\,10^8\,\frac{m}{s}$$

Section 2 – take home portion

The take home section of this exam is open book, open notes. You may also use Maple (or its equivalent) as appropriate, ***providing*** you attach a print out of the Maple worksheet. You may assume the general solution of Laplace's equation in each of the coordinate systems without needing to re-derive it. However, your answer should be your own work, collaboration is not permitted.

5. An infinitely long tube of inner radius a and outer radius b is made from a LIH material whose

dielectric constant is K. The axis of the tube is aligned with the z-axis. An external uniform electric field **E** = E_o **j** is applied. Find the electric field both inside the tube (r<a) and within its walls (a<r<b).

a. You will three potentials to solve this problem
   i. for r<a, V_{in} can contain no terms in r^{-n} which would diverge at r=0
   ii. for a<r<b, V_{mid} has all terms
   iii. for r>, V_{out} can have no terms in r^n, except the term $-E_o\, r \sin(\varphi)$ which corresponds to the applied uniform field as $r\to\infty$.

$$V_{\text{in}}=\sum a_n r^n\cos(n\phi)+\sum b_n r^n\sin(n\phi)$$

$$V_{\text{mid}}=\sum A_n r^n\cos(n\phi)+\sum B_n r^n\sin(n\phi)+\sum\frac{C_n}{r^n}\cos(n\phi)+\sum\frac{D_n}{r^n}\sin(n\phi)$$

$$V_{\text{out}}=-E_o r\sin(\phi)+\sum\frac{\alpha_n}{r^n}\cos(n\phi)+\sum\frac{\beta_n}{r^n}\sin(n\phi)$$

b. @ r=a
   i. $V_{in} = V_{mid}$
      - $a_n a^n = A_n a^n + C_n / a^n$ for all n
      - $b_n a^n = B_n a^n + D_n / a^n$ for all n
   ii. $-\partial V_{in} = -K\, \partial V_{mid}/\partial r$
      - $n\, a_n a^{n-1} = n\, A_n a^{n-1} - n\, C_n / a^{n+1}$ for all n
      - $n\, b_n a^{n-1} = n\, B_n a^{n-1} - n\, D_n / a^{n-1}$ for all n

c. @ r=b
   i. $V_{mid} = V_{out}$
      - $A_n b^n + C_n / b^n = \alpha_n / b^n$ for all n
      - $B_n b^n + D_n / b^n = \beta_n / b^n$ for all $n \neq 1$
      - $B_1 b + D_1 / b = - E_o b + \beta_1 / b$ for n = 1
   ii. $-\partial V_{in} = -K\, \partial V_{mid}/\partial r$
      - $n\, A_n b^{n-1} - n\, C_n / b^{n+1} = -n\, \alpha_n / b^{n+1}$ for all n
      - $n\, B_n b^{n-1} - n\, D_n / b^{n+1} = -n\, \beta_n / b^{n+1}$ for all $n \neq 1$
      - $B_1 - D_1 / b^2 = - E_o - \beta_1 / b^2$ for n = 1

d. Equations are solved using Maple
   i. $A_1 = B_1 = a_1 = \alpha_1 = 0$
   ii. $A_n = B_n = C_n = D_n = a_n = b_n = \alpha_n = \beta_n = 0$ for all $n \neq 1$
   iii.

$$C_1=\frac{2(1+K)E_o b^2}{-2\mathrm{K}\,b^2-2\mathrm{K}\,a^2+a^2-b^2+K^2a^2-K^2b^2}$$

$$D_1=\frac{2\mathrm{a}^2E_o b^2(K-1)}{(-2\mathrm{Kb}^2-2\mathrm{Ka}^2+a^2-b^2+K^2a^2-K^2b^2)}$$

$$b_1=\frac{4\mathrm{KE}_o b^2}{-2\mathrm{K}\,b^2-2\mathrm{K}\,a^2+a^2-b^2+K^2a^2-K^2b^2}$$

$$\beta_1=\frac{E_o b^2(-K^2b^2+K^2a^2-a^2+b^2)}{-2\mathrm{K}\,b^2-2\mathrm{K}\,a^2+a^2-b^2+K^2a^2-K^2b^2}$$

$$V_{\text{in}} = b_1 r \sin(\phi)$$
$$V_{\text{mid}} = \frac{C_1}{r}\cos(\phi) + \frac{D_1}{r}\sin(\phi)$$
$$V_{\text{out}} = -E_o r \sin(\phi) + \frac{\beta_1}{r}\sin(\phi)$$

6. A charge Q is initially at a very large distance from an infinite conducting plane. It is brought to a point which is distance a from the plane. Show that the work done is

$$W = -\frac{Q^2}{16\pi\epsilon_o a}$$

Hint: remember that the electrostatic field is a conservative field

7. A thin spherical shell of radius R holds a surface charge $\sigma = \sigma_o P_2(\cos\theta)$. Find the potential inside and outside the shell. (There is no external field.)
   a. For r<R there can be no terms in r^{-n} as they would diverge at r=0
   b. For r>R there can be no terms in r^{+n} as they would diverge as $r\rightarrow\infty$
   c. Set
   $$V_{\text{in}} = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$
   $$V_{out} = \sum_{n=0}^{\infty} \frac{b_n}{r^{n+1}} P_n(\cos\theta)$$
   d. @ r=R $V_{in} = V_{out}$
   $$A_n R^n = \frac{b_n}{R^{n+1}} \quad for\, all\, n$$
   e. @ r=R $(E_{in})_n = (E_{in})_n - \sigma$
   $$n R^{n-1} A_n = -(n+1)\frac{b_n}{R^{n+2}} \quad for\, n \neq 2$$
   $$2\text{RA}_2 = \frac{-3\text{b}_2}{R^4} + \sigma_o \quad for\, n=2$$
   f. Solving these equations
      i. A_n = b_n = 0 for n ≠ 2
      ii. $A_2 = \sigma_o / 5R$
      iii. $b_2 = \sigma_o R^4 / 5$
   g.
   $$V_{\text{in}} = \frac{\sigma_o}{5}\frac{r^2}{R} P_2(\cos\theta)$$
   $$V_{\text{out}} = \frac{\sigma_o}{5}\frac{R^4}{r^3} P_2(\cos\theta)$$

Appendix – Maple for question 5

```
eqn1 := a1*a = A1*a+B1/a; eqn5 := a1 = K*(A1-B1/a^2);

eqn2 := b1*a = C1*a+D1/a; eqn6 := b1 = K*(C1-D1/a^2);

eqn3 := A1*b+B1/b = α1/b; eqn7 := K*(A1-B1/b^2) = -α1/b^2;

eqn4 := C1*b+D1/b = -Eo*b+β1/b; eqn8 := K*(C1-D1/b^2) = -Eo-β1/b^2;

solve({eqn1, eqn2, eqn3, eqn4, eqn5, eqn6, eqn7, eqn8}, {A1, B1, C1, D1, a1, α1, b1, β1});

eqn1a := an*a^n = An*a^n+Bn/a^n; eqn5a := n*an*a^(n-1) = K*(n*An*a^(n-1)-n*Bn/a^(n+1));

eqn2a := n*bn*a^n = n*Cn*a^n+n*Dn/a^n;
eqn6a := n*bn*a^(n-1) = K*(n*Cn*a^(n-1)-n*Dn/a^(n+1));

eqn3a := An*b^n+Bn/b^n = αn/b^n; eqn7a := K*(n*An*a^(n-1)-n*Bn/b^(n+1)) = -n*αn/b^(n+1);

eqn4a := Cn*b^n+Dn/b^n = βn/b^n; eqn8a := K*n*Cn*b^(n-1)-n*Dn/b^(n+1) = -n*βn/b^(n+1);

solve({eqn1a, eqn2a, eqn3a, eqn4a, eqn5a, eqn6a, eqn7a, eqn8a}, {βn, An, Bn, Cn, Dn, an, bn, αn});
```