Laplace's Equation in two variables
Cylindrical coordinates

1. An infinitely long metal cylinder of radius R has its axis aligned with the z axis. It is placed in a uniform electric field **E** = E_o**i**. Find the electric potential and electric field everywhere.

   The general solution is

$$V_{out}=A_o\ln(r)+B_o+\sum_{n=1}^{\infty}\left(A_n r^n+\frac{B_n}{r^n}\right)\left(C_n\cos(n\varphi)+D_n\sin(n\varphi)\right)$$

   a. as $r\to\infty$ $V_{out}\to -E_o\, r\cos(\varphi)$ from which
      i. $A_o = 0$ as ln(r) diverges
      ii. $A_1 = - E_o$ n=1
      iii. $A_n = 0$ $n\neq1$
      iv. $A_nD_n = 0$ all values of n
   b. The surface of the cylinder must be an equipotential, that V_{out} is independent of φ when r = R. We can arbitrarily set this potential to be 0 without loss of accuract when calculating the field.
      i. $B_o = 0$
      ii. $B_nD_n = 0$ all values of n
      iii. $B_nC_n = 0$ $n\neq1$
      iv. $0 = -E_o R\cos(\varphi) + B_1/R\, C_1\cos(\varphi)$
         . $B_1 = E_o R^2$

$$V_{out}=-E_o r\cos(\varphi)+E_o\frac{R^2}{r}\cos(\varphi)=-E_o z+E_o\frac{R^2}{r}\cos(\varphi)$$

2. An infinitely long cylinder of radius R has its axis aligned with the z axis, and is made from a material which has a dielectric constant ε_r. It is placed in a uniform electric field **E** = E_o**i**. Find the electric potential and electric field everywhere.

   We need two general solutions

$$V_{\text{in}}=a_o\ln(r)+b_o+\sum_{n=1}^{\infty}\left(a_n r^n+\frac{b_n}{r^n}\right)\left(c_n\cos(n\varphi)+d_n\sin(n\varphi)\right)$$

$$V_{out}=A_o\ln(r)+B_o+\sum_{n=1}^{\infty}\left(A_n r^n+\frac{B_n}{r^n}\right)\left(C_n\cos(n\varphi)+D_n\sin(n\varphi)\right)$$

a. as $r \to \infty$ $V_{out} \to -E_o\, r \cos(\varphi)$ from which
   i. $A_o = 0$ as $\ln(r)$ diverges
   ii. $A_1 = -E_o$ n=1
   iii. $A_n = 0$ $n \neq 1$
   iv. $A_n D_n = 0$ all values of n
b. for r=0 V_{in} must remain finite
   i. $a_o = 0$ as $\ln(r)$ diverges
   ii. $A_1 = -E_o$ n=1
   iii. $b_n = 0$ all values of n, as r^{-n} diverges
c. for r=R $V_{in} = V_{out}$
   i. $B_o = b_o$ (they can arbitrarily be set to zero)
   ii. $a_1 R \cos(\varphi) = -E_o R\cos(\varphi) + B_1/R \cos(\varphi)$ n=1
   iii. $a_n R^n \cos(n\varphi) = B_n/R^n \cos(n\varphi)$ $n \neq 1$
d. for r=R $(D_{in})_n = (D_{out})_n$ or $\varepsilon_r \partial V_{in}/\partial r = \partial V_{out}/\partial r$
   i. $\varepsilon_r a_1 \cos(\varphi) = -E_o \cos(\varphi) + B_1/R^2 \cos(\varphi)$ n=1
   ii. $\varepsilon_r a_n n R^{n-1} \cos(n\varphi) = -n B_n/R^{n+1} \cos(n\varphi)$ $n \neq 1$
e. Solving
   i. $a_1 = -2/(\varepsilon+1) E_o$
   ii. $B_1 = (\varepsilon-1)/(\varepsilon+1) E_o R^2$

$$V_{\text{in}} = \frac{-2}{\epsilon+1} E_o r \cos(\varphi) = \frac{-2}{\epsilon+1} E_o z$$

$$V_{out} = -E_o r \cos(\varphi) + \frac{\epsilon-1}{\epsilon+1} E_o \frac{R^2}{r} \cos(\varphi)$$