Phys 3330 Electricity & Magnetism II
Spring 2017
Assignment #2 - due Wednesday February 22nd 2017

Laplace's Equation in one variable

1. Write down Laplace's equation in Cartesian coordinates if the field is known to be only dependent on on of the coordinates. Show that the field must uniform.
    a. If V = V(z) then Laplace's equation becomes $\frac{d^2V}{dz^2}=0$
    b. Integrating once $\frac{dV}{dz}=\text{constant}$
    c. Since **E** = - gradV = - dV/dz **k** the field is constant.

2. Write down Laplace's equation in spherical polar coordinates if the field is known to be only dependent on on the radius (r). Show that the field must be Coulombic.
    a. In spherical polars $\nabla^2 V=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)=0$
    b. Integrate once $r^2\frac{\partial V}{\partial r}=\text{constant}=A$
    c. rearranging $\boldsymbol{E}=-grad\,V=-\frac{\partial V}{\partial r}\hat{\boldsymbol{r}}=-\frac{A}{r^2}\hat{\boldsymbol{r}}$ which is Coulombic.

3. Write down Poisson's equation in Cartesian coordinates if the field is know to be only dependent on on of the coordinates (z). This equation is to be applied to the space between two electrodes when there is a current between them. (In tube electronics this the diode)
    a. If the space between the electrodes is evacuated then the electrons do not collided with anything, and are therefore continuously accelerated. That allows us to write the speed of the electrons (v_d) in terms of the potential, V(z).
        i. $\frac{1}{2}mv^2 = eV$
        ii. $v_d=\sqrt{\frac{2eV}{m}}$
    b. The charge density (or the number density of electrons) is not uniform in the space between the electrodes. However, Kirchoff's Law states that the current density (J) must be the same for all values of z. Use the expression $J = n(-e)v_d = -\rho v_d$ to write the charge density as a function of z.
        i. $\rho=-\frac{J}{v_d}=-\sqrt{\frac{J^2 m}{2e}}V(z)^{-1/2}=-A\,V(z)^{-1/2}$
    c. Use the result of part b to replace the charge density in Poisson's Equation, to get an ordinary differential equation in V(z). Integrate this equation once. (Hint: to integrate manually first multiply each side of the equation by 2dV/dz, or use Maple.)
        i. $\frac{d^2V}{dz^2}=\frac{-\rho}{\epsilon_o}=\frac{AV^{-1/2}}{\epsilon_o}$

ii. $$2\frac{dV}{dz}\frac{d^2V}{dz^2}=\frac{d}{dz}\left[\frac{dV}{dz}\right]^2=2\frac{dV}{dz}\frac{A}{\epsilon_o}V^{-1/2}$$

iii. $$\left[\frac{dV}{dz}\right]^2-\left[\frac{dV}{dz}\right]_o^2=\frac{4A}{\epsilon_o}V^{+1/2}$$

d. Assuming dV/dz = 0 when z = 0, integrate once more and show that the current density is inversely proportional to d^2, and directly proportional to $V_d^{3/2}$, where d is the electrode separation and V_d is the applied voltage.

i. $$\frac{dV}{dz}=\sqrt{\frac{4A}{\epsilon_o}}V^{+1/4}$$

ii. $$\int_0^{V_o}\frac{dV}{V^{+1/4}}=\int_{z=0}^{d}\sqrt{\frac{4A}{\epsilon_o}}dz$$

iii. $$\frac{4}{3}V_o^{3/4}=\sqrt{\frac{4A}{\epsilon_o}}d=\sqrt{\frac{4\sqrt{\frac{J^2m}{2e}}}{\epsilon_o}}d$$

iv. $$\frac{16}{9}V_o^{3/2}=\frac{4\sqrt{\frac{J^2m}{2e}}}{\epsilon_o}d^2$$

v. $$J=\frac{4}{9}\epsilon_o\sqrt{\frac{2e}{m}}\frac{V_o^{3/2}}{d^2}$$