Worksheet for Laplace's Equation

1. In a region where $\nabla^2 V = 0$, argue on physical grounds that minima and maxima can only occur on the boundary. (Hint: what would the electric field look like?)

Laplace's Equation in Cartesian Coordinates

2. What is the potential inside a rectangular box if the sides x=0 and y=0 are held at ground potential, and the sides x=a and y=b are held at the same fixed potential V_o? Use Maple to plot the result. (Hint: if you refer back to the example in class, and also remember the Principle of Superposition, you should be able to write down the potential inside the box without further computation.)

3. What is the potential inside a rectangular box if the sides x=0 and x=a are held at ground potential, and the sides y=0 and y=b are held at the same fixed potential V_o? Use Maple to plot the result. (Hint: if you refer back to the example in class, and also remember the Principle of Superposition, you should be able to write down the potential inside the box without further computation.)

Laplace's Equation in Cylindrical Polar Coordinates

4. Find the general solution for Laplace's Equation in cylindrical polar coordinates if $V = V(r,\varphi)$. (Use the method of separation of variables, and set the separation constant to be λ^2.)

5. An infinitely long conducting cylinder of radius R is placed in a uniform electric field $\mathbf{E} = E_o\,\mathbf{i}$. Find the electric field for all points outside the cylinder. (Remember that for a conductor the surface must be an equipotential.)
   a. Also find the charge distribution and total charge for the cylinder

Laplace's Equation in Spherical Polar Coordinates

6. Find the general solution for Laplace's Equation in spherical polar coordinates if $V = V(r,\theta)$. (Use the method of separation of variables, and set the separation constant to be $\lambda(\lambda+1)$.)

7. A conducting sphere of radius R is placed in a uniform electric field $\mathbf{E} = E_o\,\mathbf{k}$. Find the electric field for all points outside the sphere. (Remember that for a conductor the surface must be an equipotential.)
   a. Also find the charge distribution and total charge for the sphere.

8. A sphere of radius R has a surface charge distribution $\sigma_o\cos(2\theta)$. Find the potential for all points outside the sphere. (Hint: first decide what you are going to use for your boundary conditions at the surface of the sphere.)

9. Two concentric surfaces have radii a and b. (The region between them is empty.) The inner surface has a potential $P_3(\cos\theta)$ and the outer one a potential $P_5(\cos\theta)$. Find the potential between the surfaces.

10. A ring of radius R carries a uniformly distributed charge Q. If the axis of the ring is the z-axis, then symmetry dictates that $V = v(r,\theta)$.
    a. First use Coulomb's Law to find the potential along the z axis, that is $V(r,0)$.
    b. Write down the general expression for the solution to Laplace's Equation, and find the arbitrary constants by setting $\theta=0$ and comparing the expression with the answer to the first part of this question. You will first need to take the answer to the first part of this problem, using the Binomial Theorem. For $r<R$ expand in terms of r/R in order to keep the series convergent. For $r>R$ expand in terms of R/r, for the same reason.
    c. You can also get the potential $V(r,\theta)$ by using Coulomb's Law, by using the Binomial Theorem to approximate the denominator. Show that the answer is the same as you got in the second part of this question (four terms will suffice).

Laplace's Equation and Dielectrics

To solve problems involving dielectrics you will two expressions for the potential, one for inside the dielectric and one for outside the dielectric. The two expressions have the same form, but have different arbitrary constants.

There is no guarantee that the boundary of the dielectric is an equipotential. The boundary conditions to apply for the dielectric come from last semester, that the electrostatic potential, the normal component of **D**, and the tangential component of **E** are all unchanged when moving from one side of the boundary to the other. In practice you only need the first two, the last one being redundant. It yields the same equations as the first.

11. A uniform dielectric sphere of dielectric constant K_e and radius R is placed in a uniform electric field $\mathbf{E} = E_o\,\mathbf{k}$.
    a. Find the potential inside and outside the sphere.
    b. Find the electric field inside and outside the sphere. What do you notice about the field inside the sphere?
    c. Find the bound charges and the total bound charge.
    d. Find the net dipole moment of the sphere. Can you relate it to the field outside the sphere?

12. Repeat the first two parts of the previous problem for the case of a uniform dielectric cylinder of radius R placed in a uniform field $\mathbf{E} = E_o\,\mathbf{i}$.

13. A conducting shell lies between r=a and r=b (b>a). It is made from a linear isotropic material with a dielectric constant K_e. It is placed in a uniform electric field $\mathbf{E} = E_o\mathbf{k}$. Find the electric field in all three regions. (Note: for this problem you will need three potentials, all of the same mathematical form but with different constants.)

Laplace's Equation and Magnetic Materials

In the case of magnetism Laplace's Equation does not strictly speaking apply. The magnetic field is not conservative (curl**H** $\neq$ 0), and so we cannot write **H** = - grad V_m. However under conditions where we know for certain that there are no free currents then curl **H** is zero and we can write **H** in terms of a scalar magnetic potential. This is the case when there is a magnetic field imposed on the problem, generated by currents which lie outside the region of interest. It is also the case when the magnetic field is produced by a permanent magnetization rather than by an electric current. Note that this second case requires that the magnetic material must be capable of supporting a permanent magnetization, that is it must be ferromagnetic. That in turn means that the material is not linear, or isotropic, and we have to use the general expression **B** = μ_o(**H**+**M**) to relate the magnetic and induction fields.

The method for solving problems in magnetism is essentially the same as for dielectrics. At the boundary the normal component of **B**, and the tangential component of **H** are unchanged when moving from one side of the boundary to the other. There is no physical reason to assume that the scalar potential V_m is unchanged. However mathematically this condition does yield the same result as the condition that the tangential component of **H** is unchanged.

14. A ferromagnetic sphere of of radius R supports a uniform magnetization **M** = M_o**k**.
    a. Find the magnetic scalar potential inside and outside the sphere.
    b. Find the magnetic field inside and outside the sphere.

15. Repeat the first two parts of the previous problem for the case of a ferromagnetic cylinder of radius R with a uniform magnetization **M** = M_o**j**.

16. A very large ferromagnetic object has a small spherical cavity (bubble) inside it. If the magnetization in the ferromagnetic material is uniform, find the field inside the cavity.

Laplace's Equation in three variables

The extension from two variable problems to three variable problems is relatively straightforward. The differential equation is still solved using separation of variables, but the procedure has to be applied twice.

17. For a problem in Cartesian coordinates with V = V(x,y,z), find the general solution of Laplace's equation.

18. A rectangular box has five faces (x=0, y=0, z=0, y=b, and z=c) held at ground potential, and the sixth face (x=a) held at a constant potential V_o. Find the potential at all points in the box. What is the potential at the point (½ a,½ b,½ c)?